Test 3

FRI MAR 22

3.4 – 3.6, 4.1 – 4.2

(6 Questions)

Bring: Calculator

music/earplugs

Practice Problems on website

Fact: Cramer's Rule
Let A be an $n \times n$ matrix. When $\det A \neq 0$, the system $A\vec{x} = \vec{b}$ has a unique solution:
i-th variable$= \frac{|A_i|}{|A|}$
where $A_i = A$ with the i-th column replaced by $\vec{b}$.

Example: Solve using Cramer's Rule:

$$A = \begin{bmatrix} 2 & 3 & 2 \\ 3 & 0 & 5 \\ 4 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} 2x + 3y + 2z &= -11 \\ 3x \qquad + 5z &= 23 \\ 4x + \ y + \ z &= 1 \end{aligned}$$

$$A_1 = \begin{bmatrix} -11 & 3 & 2 \\ 23 & 0 & 5 \\ 1 & 1 & 1 \end{bmatrix}$$

$|A| = 47$

$|A_1| = 47$

$$|A_2| = \begin{vmatrix} 2 & -11 & 2 \\ 3 & 23 & 5 \\ 4 & 1 & 1 \end{vmatrix}$$

(along Row 1)

$$= 2 \begin{vmatrix} 23 & 5 \\ 1 & 1 \end{vmatrix} + 11 \begin{vmatrix} 3 & 5 \\ 4 & 1 \end{vmatrix} + 2 \begin{vmatrix} 3 & 23 \\ 4 & 1 \end{vmatrix}$$

$$= 2(18) + 11(-17) + 2(-89)$$

$$= -329$$

$$|A_3| = \begin{vmatrix} 2 & 3 & -11 \\ 3 & 0 & 23 \\ 4 & 1 & 1 \end{vmatrix}$$

(along Column 2)

$$= -3 \begin{vmatrix} 3 & 23 \\ 4 & 1 \end{vmatrix} - 1 \begin{vmatrix} 2 & -11 \\ 3 & 23 \end{vmatrix}$$

$$= -3(-89) - (79)$$

$$= 188$$

$$x = \frac{|A_1|}{|A|} = 1$$

$$y = \frac{|A_2|}{|A|} = -7$$

$$z = \frac{|A_3|}{|A|} = 4$$

Definition: A **cofactor** is the signed determinant in the cofactor expansion that's associated with a matrix entry. It's written C_{ij}.

The sign is given by the checkerboard pattern: $\begin{bmatrix} + & - & + & \dots \\ - & + & - & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$.

Example: Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 5 \end{bmatrix}$.
Calculate the cofactors C_{11}, C_{12} and C_{32}.

$$C_{11} = +\begin{vmatrix} 0 & 3 \\ 1 & 5 \end{vmatrix} = +(-3) = -3$$

$$C_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = -2$$

$$C_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2$$

Definition: The **cofactor matrix** is the matrix whose entries are the cofactors of A.

Example: Find the cofactor matrix for $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 5 \end{bmatrix}$.

$$\text{Cofactor Matrix} = \begin{bmatrix} -3 & -2 & 1 \\ 11 & 4 & -3 \\ -6 & -2 & 2 \end{bmatrix}$$

Definition: The **adjoint of** A is the transpose of the cofactor matrix. It's written $\text{adj}(A)$.

Fact: For an $n \times n$ matrix with $|A| \neq 0$:
$A^{-1} = \frac{1}{|A|}\,\text{adj}(A)$.

Example: Let $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 2 \\ 3 & -3 & 4 \end{bmatrix}$. Find A^{-1} using the adjoint formula.

$$\text{Cofactor Matrix} = \begin{bmatrix} 10 & -2 & -9 \\ -14 & 2 & 12 \\ 2 & 0 & -2 \end{bmatrix}$$

transpose

$$\text{adj}(A) = \begin{bmatrix} 10 & -14 & 2 \\ -2 & 2 & 0 \\ -9 & 12 & -2 \end{bmatrix}$$

(along Row 1)

$$|A| = 2\begin{vmatrix} 1 & 2 \\ -3 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 3 & -3 \end{vmatrix}$$

$$= 2(10) - 2(2) + 2(-9)$$

$$= -2$$

$$A^{-1} = \frac{-1}{2}\,\text{adj}(A)$$

Example: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find A^{-1} using the adjoint formula.

Cofactor Matrix $= \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

$$\operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$|A| = ad - bc$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

4.3 Eigenvalues and Eigenvectors, $n \times n$ Matrices

Example: Find all the eigenvalues of $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -4 & 3 \\ 0 & 0 & 7 \end{bmatrix}$.

Solve $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & -4-\lambda & 3 \\ 0 & 0 & 7-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-4-\lambda)(7-\lambda) = 0$$

because the matrix is upper triangular

$$\lambda = 1, -4, 7$$

Fact: The eigenvalues of an upper triangular, lower triangular or diagonal matrix are the diagonal entries.

Integer Roots Theorem
If a polynomial has integer coefficients and the leading coefficient is 1 then any integer roots divide the constant.

Example: Find all the eigenvalues of $A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$.

Solve $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ 7 & -5-\lambda & 1 \\ 6 & -6 & 2-\lambda \end{vmatrix} = 0$$

Expand along Row 1:

$$(3-\lambda)\begin{vmatrix} -5-\lambda & 1 \\ -6 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 7 & 1 \\ 6 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 7 & -5-\lambda \\ 6 & -6 \end{vmatrix} = 0$$

$$(3-\lambda)\left[(-5-\lambda)(2-\lambda)+6\right] + \left[7(2-\lambda)-6\right] + \left[-42-6(-5-\lambda)\right] = 0$$

$$(3-\lambda)\left[\lambda^2+3\lambda-4\right] + \left[-7\lambda+8\right] + \left[6\lambda-12\right] = 0$$

Example Continued...

$$3\lambda^2 + 9\lambda - 12$$

$$-\lambda^3 - 3\lambda^2 + 4\lambda$$

$$-7\lambda + 8$$

$$6\lambda - 12$$

$$-\lambda^3 + 12\lambda - 16 = 0$$

or $$\lambda^3 - 12\lambda + 16 = 0$$