Fact: Let $\vec{u}$ and $\vec{v}$ be in $\mathbb{R}^n$. The angle θ between $\vec{u}$ and $\vec{v}$ is defined to be $0° \leq \theta \leq 180°$

Fact: For all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$: $\vec{u} \cdot \vec{v} = ||\vec{u}|| \; ||\vec{v}|| \cos\theta$

Comment: In $\mathbb{R}^4$ and higher dimensions, this is a definition of θ.

Comment: In the special case where $\vec{u}$ and $\vec{v}$ are unit vectors, $\vec{u} \cdot \vec{v}$ gives the value of $\cos\theta$.

Example: Find the angle between $\vec{u} = [1, -4]$ and $\vec{v} = [2, 3]$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$

$$-10 = \sqrt{17}\sqrt{13} \cos\theta$$

$$\frac{-10}{\sqrt{17}\sqrt{13}} = \cos\theta$$

$$\theta = \cos^{-1}\left(\frac{-10}{\sqrt{17}\sqrt{13}}\right) \approx 132^\circ$$

Example: If $0^\circ \leq \theta < 90^\circ$, what is the sign of $\vec{u} \cdot \vec{v}$?
What if $\theta = 90^\circ$?
What if $90^\circ < \theta \leq 180^\circ$?

$0^\circ \leq \theta < 90^\circ$
$\Leftrightarrow \cos\theta > 0$
$\Leftrightarrow \vec{u} \cdot \vec{v} > 0$

$\theta = 90^\circ$
$\Leftrightarrow \cos\theta = 0$
$\Leftrightarrow \vec{u} \cdot \vec{v} = 0$

$90^\circ < \theta \leq 180^\circ$
$\Leftrightarrow \cos\theta < 0$
$\Leftrightarrow \vec{u} \cdot \vec{v} < 0$

Definition: Vectors $\vec{u}$ and $\vec{v}$ are **orthogonal** if $\vec{u} \cdot \vec{v} = 0$.

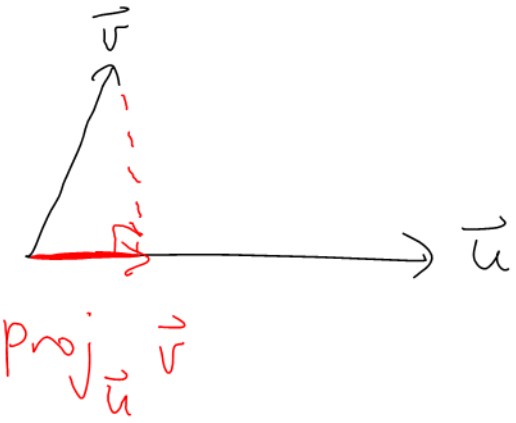
Comment: The following statements are equivalent in 2D and 3D:
Vectors $\vec{u}$ and $\vec{v}$ are perpendicular (geometry language)
Vectors $\vec{u}$ and $\vec{v}$ are orthogonal (algebra language)

Comment: In higher dimensions it's more appropriate to use the word **orthogonal** rather than perpendicular.

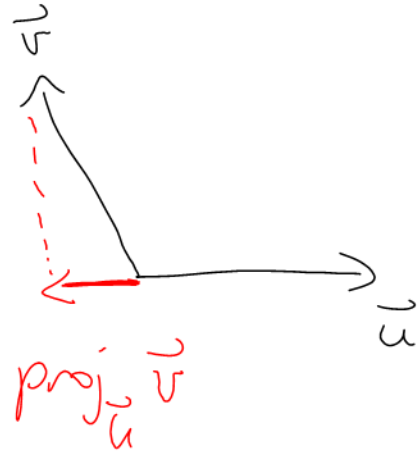
Definition: The **projection** of $\vec{v}$ onto $\vec{u}$ is written $\text{proj}_{\vec{u}}\vec{v}$. This could be read as the projection onto $\vec{u}$ of $\vec{v}$.

Example: Let's draw a few instances of $\text{proj}_{\vec{u}}\vec{v}$

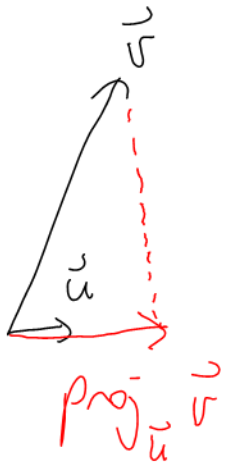
a)



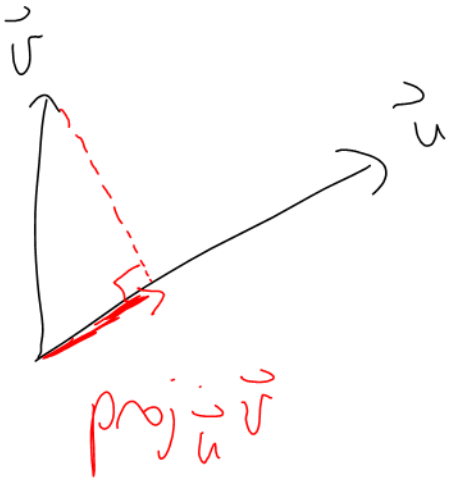
b)



c)



d)



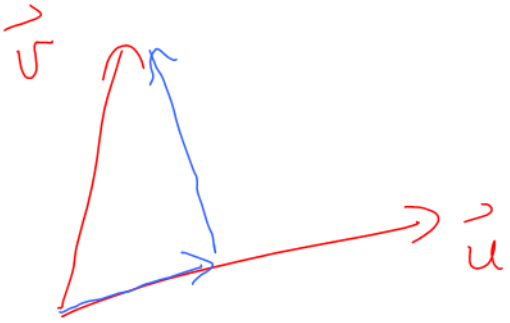
Fact: $\text{proj}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||^2}\vec{u}$

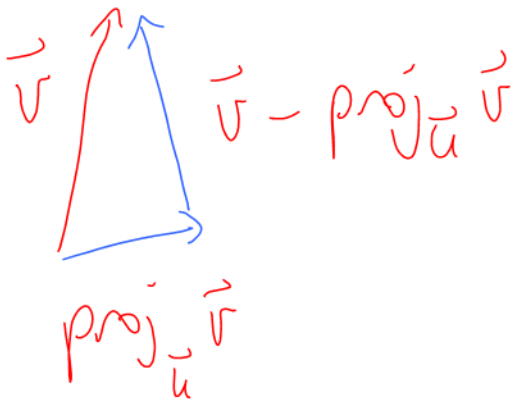
Example: Find $\text{proj}_{\vec{u}}\vec{v}$ for $\vec{u} = [1, 2]$ and $\vec{v} = [1, 3]$

$$\text{proj}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\vec{u}$$

$$= \frac{7}{5}\vec{u} \quad \text{or} \quad \frac{7}{5}[1, 2]$$

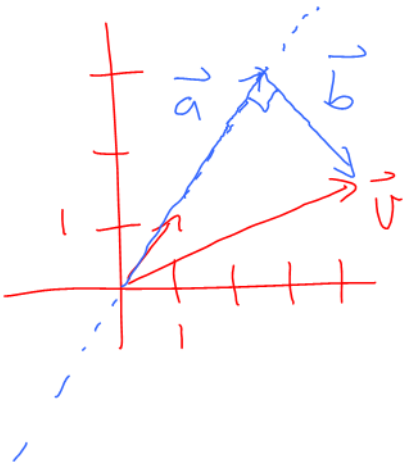
Fact: Given vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^n$, there is exactly one way to decompose $\vec{v}$ into two vectors that are parallel and perpendicular to $\vec{u}$.





Example: Let $\vec{u} = [1, 1]$ and $\vec{v} = [4, 2]$. Find vectors $\vec{a}$ and $\vec{b}$ so that $\vec{v} = \vec{a} + \vec{b}$, $\vec{a}$ is parallel to $\vec{u}$, and $\vec{b}$ is perpendicular to $\vec{u}$.

Roughly



$$\vec{a} = \text{proj}_{\vec{u}} \vec{v}$$

$$= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u}$$

$$= \frac{6}{2} [1, 1]$$

$$= 3 [1, 1]$$

$$= [3, 3]$$

$$\vec{v} = \vec{a} + \vec{b}$$

$$\vec{b} = \vec{v} - \vec{a}$$

$$= [4, 2] - [3, 3]$$

$$= [1, -1]$$