Example: Calculate $|A|$ for $A = \begin{bmatrix} 1 & 6 & 2 & 3 \\ 0 & 0 & 0 & 4 \\ 2 & 1 & 1 & 6 \\ 2 & 0 & 5 & 7 \end{bmatrix}$.

Example: In this example we'll illustrate the **Quick Method** for 3×3 Determinants. Calculate $\det A$ using the Quick Method. Let $A = \begin{bmatrix} 1 & 4 & 9 \\ 2 & -2 & 6 \\ 1 & 0 & 4 \end{bmatrix}$.

$$\begin{array}{ccccc} & & - & - & - \\ 1 & 4 & 9 & 1 & 4 \\ 2 & -2 & 6 & 2 & -2 \\ 1 & 0 & 4 & 1 & 0 \end{array}$$

$$18 \quad 0 \quad -32 \quad -8 \quad 24 \quad 0$$

$$|A| = 18 + 0 - 32 - 8 + 24 + 0$$
$$= 2$$

Comment: The Quick Method only applies for 3×3 matrices.

Fact: If A is upper triangular, lower triangular or diagonal then $\det A$ is the product of the diagonal entries. For example $\det \begin{bmatrix} 2 & 9 & 13 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{bmatrix} = -8.$

$$\begin{vmatrix} 2 & 0 & 0 \\ 9 & 3 & 0 \\ 9 & 9 & 7 \end{vmatrix} = 42$$

$$\begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{vmatrix} = 42$$

Example: Let's understand why by calculating $\det \begin{bmatrix} 2 & 9 & 13 \\ 0 & -1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$.

$$= 4 \det \begin{bmatrix} 2 & 9 \\ 0 & -1 \end{bmatrix}$$

$$= 4 \left[2(-1) \right]$$

$$= -8$$

Fact: How Row Operations Change the Determinant:

$R_i \pm kR_j$ does not change the determinant.

$R_i \leftrightarrow R_j$ changes the sign of the determinant.

We can factor any row, for example $\det \begin{bmatrix} 3 & 6 \\ 1 & 5 \end{bmatrix} = 3 \det \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix}$.

Example: Calculate the determinant by reducing the matrix to REF.

Let $A = \begin{bmatrix} 1 & -2 & 1 & 9 \\ 2 & 1 & 3 & 3 \\ 3 & 1 & 4 & 5 \\ 0 & 1 & 1 & 6 \end{bmatrix}$.

$\boxed{\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}}$

$$|A| = \begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 5 & 1 & -15 \\ 0 & 7 & 1 & -22 \\ 0 & 1 & 1 & 6 \end{vmatrix}$$

$\boxed{R_2 \Leftrightarrow R_4}$

$$= - \begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 1 & 1 & 6 \\ 0 & 7 & 1 & -22 \\ 0 & 5 & 1 & -15 \end{vmatrix}$$

$\boxed{\begin{array}{l} R_3 \to R_3 - 7R_2 \\ R_4 \to R_4 - 5R_2 \end{array}}$

$$= - \begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -6 & -64 \\ 0 & 0 & -4 & -45 \end{vmatrix}$$

$\boxed{\frac{R_3}{-6}}$

$$= -(-6) \begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & \frac{64}{6} \\ 0 & 0 & -4 & -45 \end{vmatrix}$$

$\boxed{R_4 \to R_4 + 4R_3}$

$$= 6 \begin{vmatrix} 1 & -2 & 1 & 9 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & \frac{64}{6} \\ 0 & 0 & 0 & -\frac{7}{3} \end{vmatrix}$$

$$= 6 \left[1 \cdot 1 \cdot 1 \cdot -\frac{7}{3} \right]$$

$$= -14$$

Comment: In general $\det A \neq \det(\text{REF of } A)$.

Fact: An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$.

Fact: Properties of $\det A$:

1) $\det A^{-1} = \frac{1}{\det A}$ (if $\det A \neq 0$)

2) $\det AB = \det A \cdot \det B$

3) $\det kA = k^n \det A$ (where A is $n \times n$) !

4) $\det A^T = \det A$

Comment: To illustrate Property 3:
$\det \begin{bmatrix} 7a & 7b \\ 7c & 7d \end{bmatrix} = 7^2 \det \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

$$\det \begin{bmatrix} 5a & 5b & 5c \\ 5d & 5e & 5f \\ 5g & 5h & 5i \end{bmatrix} = 5^3 \det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Comment: Note that $\det(A + B) \neq \det A + \det B$ in general.

Example: Let $\det A \neq 0$. Prove Property 1.

$$\det A \neq 0$$
$$\Rightarrow A^{-1} \text{ exists}$$
$$\Rightarrow A^{-1}A = I$$
$$\Rightarrow \det(A^{-1}A) = \det I$$
$$\Rightarrow \det A^{-1} \cdot \det A = 1$$
$$\Rightarrow \det A^{-1} = \frac{1}{\det A}$$

Fact: Cramer's Rule
Let A be an $n \times n$ matrix. When $\det A \neq 0$, the system $A\vec{x} = \vec{b}$ has a unique solution:
i-th variable= $\frac{|A_i|}{|A|}$
where $A_i = A$ with the i-th column replaced by $\vec{b}$.

Example: Solve using Cramer's Rule:

$$A = \begin{bmatrix} 2 & 3 & 2 \\ 3 & 0 & 5 \\ 4 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} 2x + 3y + 2z &= -11 \\ 3x + 5z &= 23 \\ 4x + y + z &= 1 \end{aligned}$$

$$A_2 = \begin{bmatrix} 2 & -11 & 2 \\ 3 & 23 & 5 \\ 4 & 1 & 1 \end{bmatrix}$$

2nd row:

$$|A| = -3\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} - 5\begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = -3(1) - 5(-10) = 47$$

1st row:

$$|A_2| = 2\begin{vmatrix} 23 & 5 \\ 1 & 1 \end{vmatrix} + 11\begin{vmatrix} 3 & 5 \\ 4 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 23 \\ 4 & 1 \end{vmatrix}$$

$$= 2(18) + 11(-17) + 2(-89)$$

$$= -329$$

$$y = \frac{|A_2|}{|A|}$$

$$= -7$$

x and z on Monday