Fact: Let B be an $n \times n$ matrix. The system $B\vec{x} = \vec{0}$ has nontrivial solutions exactly when $\det B = 0$. (This follows from the Fundamental Theorem of Invertible Matrices). (Section 3.5)

Fact: To find all the eigenvalues of A: Solve the equation $\det(A - \lambda I) = 0$.

Example: Let's understand why solving $\det(A - \lambda I) = 0$ gives the eigenvalues.

λ is an eigenvalue of A

$\iff$ $A\vec{x} = \lambda\vec{x}$ for $\vec{x} \neq \vec{0}$

$\iff$ $(A - \lambda I)\vec{x} = \vec{0}$ for $\vec{x} \neq \vec{0}$

$\iff$ $\det(A - \lambda I) = 0$

Example: Find all the eigenvalues of $A = \begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix}$.

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 4-\lambda & -2 \\ 5 & -7-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(-7-\lambda) - (-10) = 0$$

$$-28 - 4\lambda + 7\lambda + \lambda^2 + 10 = 0$$

$$\lambda^2 + 3\lambda - 18 = 0$$

$$(\lambda + 6)(\lambda - 3) = 0$$

$$\lambda = -6, 3$$

Example: Find a basis for E_{-6} given $A = \begin{bmatrix} 4 & -2 \\ 5 & -7 \end{bmatrix}$.

$\lambda = -6$

$$[A - \lambda I \mid \vec{0}]$$

$$[A + 6I \mid \vec{0}]$$

$$\left[\begin{array}{cc|c} 10 & -2 & 0 \\ 5 & -1 & 0 \end{array}\right]$$

$$\frac{R_1}{10} \quad \left[\begin{array}{cc|c} 1 & -\frac{1}{5} & 0 \\ 5 & -1 & 0 \end{array}\right]$$

$$R_2 - 5R_1 \quad \begin{array}{cc} x_1 & x_2 \end{array} \left[\begin{array}{cc|c} 1 & -\frac{1}{5} & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

$x_2 = t$

$x_1 - \frac{1}{5}x_2 = 0 \Rightarrow x_1 = \frac{1}{5}t$

eigenvectors $\vec{x} = \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} t \quad (t \neq 0)$

basis for $E_{-6} = \left\{ \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right\}$

Comment:

To find eigenvalues: Solve the equation $\det(A - \lambda I) = 0$.

To find eigenvectors: Solve the system $[A - \lambda I \mid \vec{0}]$. Remember to exclude $\vec{x} = \vec{0}$.

Example: Let $A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$. Find the eigenvectors and eigenvalues geometrically.

In a problem like this, the eigenvectors will be parallel to the x-axis and the y-axis.

Let $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$
$$= -2\vec{x}$$

Let $\vec{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$= 1\vec{x}$$

$\boxed{\lambda = -2}$ $E_{-2} = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

OR A basis for $E_{-2} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$

OR $\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t \quad (t \neq 0)$

$\boxed{\lambda = 1}$ $E_1 = \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$

4.2 Determinants

Comment: Recall that the determinant of A is written $\det A$ or $|A|$. It's only defined for square matrices.

Fact: The cofactor expansion from Section 1.4 generalizes according to the following rules:

We can expand along any row or column.

The sign associated with each term follows the checkerboard pattern: $\begin{bmatrix} + & - & + & \dots \\ - & + & - & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$.

Example: Find $\det A$ by cofactor expansion along the second column. Calculate it again by cofactor expansion along the third row. Let $A = \begin{bmatrix} 4 & 1 & 6 \\ 1 & 2 & 3 \\ 6 & 0 & 7 \end{bmatrix}$.

$$\begin{bmatrix} + & - & \\ & + & \\ & - & \end{bmatrix}$$

2nd COLUMN:

$$\det A = -1\left|\quad\right| + 2\left|\quad\right| - 0\left|\quad\right|$$

$$= -1\begin{vmatrix} 1 & 3 \\ 6 & 7 \end{vmatrix} + 2\begin{vmatrix} 4 & 6 \\ 6 & 7 \end{vmatrix}$$

$$= -1(-11) + 2(-8)$$

$$= -5$$

$$\begin{bmatrix} + & & \\ - & & \\ + & - & + \end{bmatrix}$$

3rd ROW:

$$\det A = 6\begin{vmatrix} 1 & 6 \\ 2 & 3 \end{vmatrix} + 7\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix}$$

$$= 6(-9) + 7(7)$$

$$= -5$$

Example: Calculate $|A|$ for $A = \begin{bmatrix} 1 & 6 & 2 & 3 \\ 0 & 0 & 0 & 4 \\ 2 & 1 & 1 & 6 \\ 2 & 0 & 5 & 7 \end{bmatrix}$.

$$\begin{bmatrix} + & & & - \\ - & + & - & + \\ & & & \end{bmatrix}$$

2nd Row:

$$|A| = 4 \begin{vmatrix} 1 & 6 & 2 \\ 2 & 1 & 1 \\ 2 & 0 & 5 \end{vmatrix}$$

$$= 4\left[2\begin{vmatrix} 6 & 2 \\ 1 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 6 \\ 2 & 1 \end{vmatrix}\right]$$

$$= 4[2(4) + 5(-11)]$$

$$= -188$$

Example: In this example we'll illustrate the **Quick Method** for 3×3 Determinants. Calculate $\det A$ using the Quick Method. Let $A = \begin{bmatrix} 1 & 4 & 9 \\ 2 & -2 & 6 \\ 1 & 0 & 4 \end{bmatrix}$.

Comment: The Quick Method only applies for 3×3 matrices.