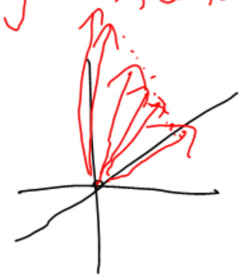
Definition: Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$.
The **inverse of** T is a transformation $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ such that:
$T^{-1}(T(\vec{x})) = \vec{x}$ and $T(T^{-1}(\vec{x})) = \vec{x}$ for all vectors $\vec{x}$ in $\mathbb{R}^n$.

Can't undo projections



Comment: Note that T^{-1} is only defined when $[T]$ is invertible.

Fact: The standard matrix for T^{-1} is the inverse of the standard matrix for T.

Example: Rewrite this fact using appropriate notation.

$$[T^{-1}] = [T]^{-1}$$

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation by $-30°$. Find $[T^{-1}]$.

Method 1

$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta = -30^\circ$$

$$= \begin{bmatrix} \sqrt{3}/2 & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$[T^{-1}] = [T]^{-1}$$

$$\det[T] = \frac{3}{4} + \frac{1}{4} = 1$$

$$[T^{-1}] = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

Method 2:

T^{-1}: rotation by $30°$

$$[T^{-1}] = \begin{bmatrix} \sqrt{3}/2 & -\frac{1}{2} \\ \frac{1}{2} & \sqrt{3}/2 \end{bmatrix}$$

$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(c\vec{u}) = cT(\vec{u})$

Example: Let T be a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$. Suppose:

$\vec{v}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}$ and $T(\vec{v}_1) = \begin{bmatrix}-5\\8\end{bmatrix}$,

$\vec{v}_2 = \begin{bmatrix}1\\0\\1\end{bmatrix}$ and $T(\vec{v}_2) = \begin{bmatrix}2\\2\end{bmatrix}$, and

$\vec{v}_3 = \begin{bmatrix}0\\1\\1\end{bmatrix}$ and $T(\vec{v}_3) = \begin{bmatrix}-1\\3\end{bmatrix}$.

Find $T(\begin{bmatrix}7\\3\\6\end{bmatrix})$.

Let $\begin{bmatrix}7\\3\\6\end{bmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$

$$\begin{array}{cccc} c_1 & c_2 & c_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 6 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$c_1 = 2 \quad c_2 = 5 \quad c_3 = 1$

$$\begin{aligned} T(\begin{bmatrix}7\\3\\6\end{bmatrix}) &= T(2\vec{v}_1 + 5\vec{v}_2 + \vec{v}_3) \\ &= T(2\vec{v}_1) + T(5\vec{v}_2) + T(\vec{v}_3) \\ &= 2T(\vec{v}_1) + 5T(\vec{v}_2) + T(\vec{v}_3) \\ &= 2\begin{bmatrix}-5\\8\end{bmatrix} + 5\begin{bmatrix}2\\2\end{bmatrix} + \begin{bmatrix}-1\\3\end{bmatrix} \\ &= \begin{bmatrix}-1\\29\end{bmatrix} \end{aligned}$$

Chapter 4: Eigenvalues and Eigenvectors

4.1 Eigenvalues and Eigenvectors, 2×2 Matrices

Definition: Let A be an $n \times n$ matrix. Suppose $A\vec{x} = \lambda\vec{x}$ for some vector $\vec{x} \neq \vec{0}$ and some real number λ. Then λ is an **eigenvalue of** A and $\vec{x}$ is an **eigenvector of** A.

Example: Show that $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix}$.

$$A\vec{x} = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

$$= 4 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= 4\vec{x}$$

Comment: We say that $\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector of A corresponding to eigenvalue $\lambda = 4$.

Comment: Note that $A\vec{0} = \lambda\vec{0}$ is trivial. Therefore the zero vector is never considered to be an eigenvector.

$$A\vec{0} = \vec{0}$$

$$= \lambda\vec{0} \qquad \text{obvious}$$

Example: Find all eigenvectors of $A = \begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix}$ corresponding to $\lambda = 6$.

Want $A\vec{x} = 6\vec{x}$

$$A\vec{x} = 6I\vec{x}$$

$$A\vec{x} - 6I\vec{x} = \vec{0}$$

$$(A - 6I)\vec{x} = \vec{0}$$

Solve $[A - 6I \mid \vec{0}]$

shortcut

$$A - 6I = \begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix} - 6\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -3 & 4 \end{bmatrix} + \begin{bmatrix} -6 & 0 \\ 0 & -6 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ -3 & -2 \end{bmatrix}$$

shortcut

$$[A - 6I \mid \vec{0}]$$

$$\left[\begin{array}{cc|c} -3 & -2 & 0 \\ -3 & -2 & 0 \end{array}\right]$$

$\frac{R_1}{-3}$ $\left[\begin{array}{cc|c} 1 & \frac{2}{3} & 0 \\ -3 & -2 & 0 \end{array}\right]$

$R_2 + 3R_1$ $\begin{array}{ccc} x_1 & x_2 & \\ \end{array}$ $\left[\begin{array}{cc|c} 1 & \frac{2}{3} & 0 \\ 0 & 0 & 0 \end{array}\right]$ RREF

$$x_2 = t$$

$$x_1 + \tfrac{2}{3}x_2 = 0 \Rightarrow x_1 = -\tfrac{2}{3}t$$

$\vec{x} = \begin{bmatrix} -2/3 \\ 1 \end{bmatrix} t \quad (t \neq 0)$ OR $\vec{x} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} t \quad (t \neq 0)$

Check: $A\begin{bmatrix} 2 \\ -3 \end{bmatrix} = 6\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ ✓

Fact: To find all the eigenvectors corresponding to eigenvalue λ:
Solve the system $[A - \lambda I \mid \vec{0}]$. Remember to exclude $\vec{x} = \vec{0}$.

Definition: The **eigenspace** E_λ is the set of all eigenvectors of A corresponding to eigenvalue λ, plus the zero vector. It's a subspace of $\mathbb{R}^n$

$E_\lambda = \{\text{eigenvectors corresponding to } \lambda\} \cup \{\vec{0}\}$

line through origin or
plane through origin etc.

Example: Find a basis for E_3 given $A = \begin{bmatrix} 4 & 1 & -2 \\ -3 & 0 & 6 \\ 2 & 2 & -1 \end{bmatrix}$.

$$[A - \lambda I \mid \vec{0}]$$

$$\lambda = 3: \quad [A - 3I \mid \vec{0}]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ -3 & -3 & 6 & 0 \\ 2 & 2 & -4 & 0 \end{array}\right]$$

$$\begin{array}{c} R_2 + 3R_1 \\ R_3 - 2R_1 \end{array} \quad \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \text{RREF}$$

$x_2 = s$

$x_3 = t$

$$x_1 + x_2 - 2x_3 = 0 \implies x_1 = -s + 2t$$

$$\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} t \qquad (\vec{x} \neq \vec{0})$$

$$\text{Basis for } E_3 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

E_3

Example: Find a basis for E_0 given $A = \begin{bmatrix} 4 & 1 & -3 \\ 0 & 0 & 2 \\ 0 & 0 & -3 \end{bmatrix}$.

0 can be an eigenvalue

$$[A - \lambda I \mid \vec{0}]$$

$$[A \mid \vec{0}]$$

$$\left[\begin{array}{ccc|c} 4 & 1 & -3 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \text{RREF}$$

$x_2 = t$

$$x_1 + \frac{1}{4}x_2 = 0 \implies x_1 = -\frac{1}{4}t$$

$$x_3 = 0$$

$$\vec{x} = \begin{bmatrix} -1/4 \\ 1 \\ 0 \end{bmatrix} t \quad \text{or} \quad \vec{x} = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} t \quad (t \neq 0)$$

$$\text{Basis for } E_0 = \left\{ \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} \right\}$$