Definition: The **standard matrix for** T is the matrix that performs T. It's written $[T]$.

Fact: Let $T : \mathbb{R}^2 \to \mathbb{R}^m$ be a linear transformation. To calculate $[T]$:

Place $T(\begin{bmatrix}1\\0\end{bmatrix})$ in the first column and place $T(\begin{bmatrix}0\\1\end{bmatrix})$ in the second column.

In other words, $[T] = \left[T(\begin{bmatrix}1\\0\end{bmatrix}) \;\middle|\; T(\begin{bmatrix}0\\1\end{bmatrix}) \right]$.

Fact: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then:

$$[T] = \left[T(\begin{bmatrix}1\\0\\0\\\cdots\\0\end{bmatrix}) \;\middle|\; T(\begin{bmatrix}0\\1\\0\\\cdots\\0\end{bmatrix}) \;\middle|\; \cdots \;\middle|\; T(\begin{bmatrix}0\\0\\0\\\cdots\\1\end{bmatrix}) \right].$$

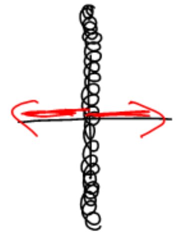
$T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ and $T(c\vec{u}) = c\,T(\vec{u})$

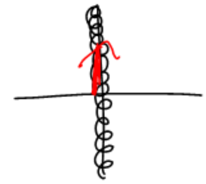
Comment: The formula for $[T]$ works because T is linear.

$$T(\begin{bmatrix}x\\y\end{bmatrix})$$
$$= T(x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix})$$
$$= T(x\begin{bmatrix}1\\0\end{bmatrix}) + T(y\begin{bmatrix}0\\1\end{bmatrix})$$
$$= x\,T(\begin{bmatrix}1\\0\end{bmatrix}) + y\,T(\begin{bmatrix}0\\1\end{bmatrix})$$
$$= \underbrace{\begin{bmatrix} T(\begin{bmatrix}1\\0\end{bmatrix}) & T(\begin{bmatrix}0\\1\end{bmatrix}) \end{bmatrix}}_{\text{standard matrix for } T} \begin{bmatrix}x\\y\end{bmatrix}$$

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that reflects a vector in the y-axis. Find:

a) $[T]$





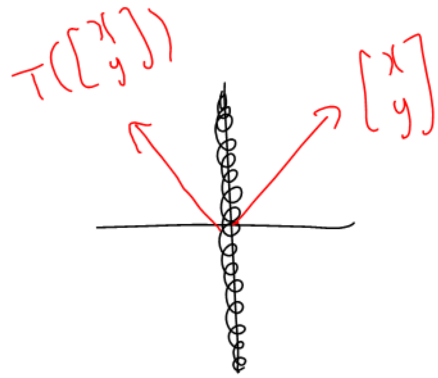
$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

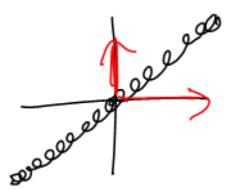
$$[T] = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$$

b) $T(\begin{bmatrix}x\\y\end{bmatrix})$ $= \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$

$= \begin{bmatrix}-x\\y\end{bmatrix}$



Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that reflects a vector in the line $y = x$. Find $[T]$.



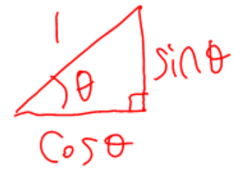
$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$$

$$[T] = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates a vector by angle θ (counterclockwise). Find $[T]$.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix}$$



$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

$$[T] = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

⊗ Know this

Example: Rotate $\begin{bmatrix}1\\1\end{bmatrix}$ by $30°$ clockwise.

$$\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix} \quad \theta = -30°$$

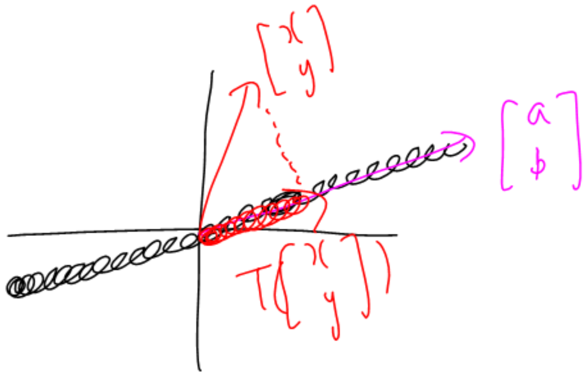
$$= \frac{1}{2}\begin{bmatrix}\sqrt{3} & 1\\ -1 & \sqrt{3}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}\sqrt{3}+1\\ -1+\sqrt{3}\end{bmatrix}$$

$\sin(-30°) = -\frac{1}{2}$

$\cos(-30°) = \frac{\sqrt{3}}{2}$

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that projects a vector on the line l through the origin with direction vector $\vec{d} = \begin{bmatrix} a \\ b \end{bmatrix}$. Find $[T]$.



$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \text{proj}_{\begin{bmatrix} a \\ b \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{\begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} a \\ b \end{bmatrix} \right\|^2} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \frac{a}{a^2+b^2} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$= \frac{1}{a^2+b^2} \begin{bmatrix} a^2 \\ ab \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \text{proj}_{\begin{bmatrix} a \\ b \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{b}{a^2+b^2} \begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{a^2+b^2} \begin{bmatrix} ab \\ b^2 \end{bmatrix}$$

$$[T] = \frac{1}{a^2+b^2} \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$$ Know this

Comment: It's recommended that you know the following two standard matrices:

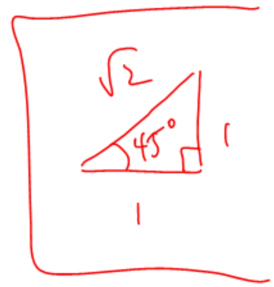
Rotation by angle θ (counterclockwise): $[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Projection onto the line $\vec{x} = t \begin{bmatrix} a \\ b \end{bmatrix}$: $[T] = \frac{1}{a^2+b^2} \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix}$

Definition: Suppose we apply T_1 then T_2 to $\vec{x}$. This is a **composition** of transformations. It can be written $T_2(T_1(\vec{x}))$ or $(T_2 \circ T_1)(\vec{x})$.
We calculate it as $[T_2]\,[T_1]\,\vec{x}$.

Example: Let $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x \\ -y \end{bmatrix}$. Let $S : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation by $45°$. Find $[S \circ T]$.

$$[S] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta = 45° \quad = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

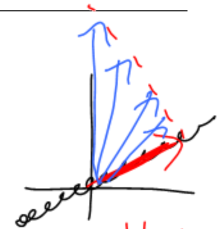


$$[T] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

coefficients

$$[S][T] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 1 & 0 \\ 2 & -1 & 0 \end{bmatrix}$$



Can't undo projections.

Definition: Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$.
The **inverse of** T is a transformation $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ such that:
$T^{-1}(T(\vec{x})) = \vec{x}$ and $T(T^{-1}(\vec{x})) = \vec{x}$ for all vectors $\vec{x}$ in $\mathbb{R}^n$.

Comment: Note that T^{-1} is only defined when $[T]$ is invertible.

Fact: The standard matrix for T^{-1} is the inverse of the standard matrix for T.

Example: Rewrite this fact using appropriate notation.

$$[T^{-1}] = [T]^{-1}$$

Example: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a rotation by $-30°$. Find $[T^{-1}]$.