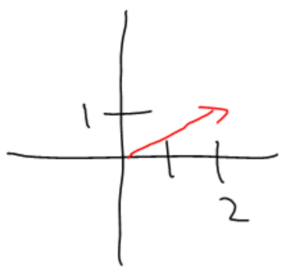
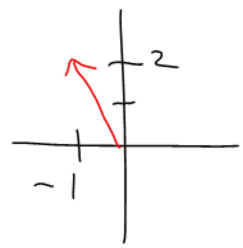
3.6 Linear Transformations

Definition: A **transformation** is an operation that turns a vector into another vector.

Example: The transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ rotates a vector by 90° counterclockwise. Graph the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ before and after the transformation.





Definition: The vector $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ is called the **image of** $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ **under** T .

We can write $T(\begin{bmatrix} 2 \\ 1 \end{bmatrix}) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ or $T\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

Definition: The **matrix transformation** T_A multiplies a vector on the left by matrix A. In other words, $T_A(\vec{x}) = A\vec{x}$.

Example: a) Let $A = \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & -3 \end{bmatrix}$. Find $T_A(\begin{bmatrix} x \\ y \\ z \end{bmatrix})$.

$$= A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ -1 & 1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= \begin{bmatrix} 2x + z \\ -x + y - 3z \end{bmatrix}$$

b) Find A given $T_A(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 2x + y \\ x - y \\ 3x + 3y \end{bmatrix}$.

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x + y \\ x - y \\ 3x + 3y \end{bmatrix}$$

Coefficients

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 3 \end{bmatrix}$$

Definition: A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is **linear** if:

* $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ and

* $T(c\vec{u}) = cT(\vec{u})$ for all real numbers c and all vectors $\vec{u}$ in $\mathbb{R}^n$.

2 nice properties

$T(\vec{x}) = A\vec{x}$

Fact: The transformation T is linear if and only if T is a matrix transformation.

Example: Show that T is linear given $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$.

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

T is a matrix transformation

$\Rightarrow$ T is linear

Example: Show that T is not linear given $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ 1+x \end{bmatrix}$.

$$\begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1+x \end{bmatrix}$$

No matrix M exists so that

$$M\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 1+x \end{bmatrix}$$

Note: M can contain numbers but not variables.

T is not a matrix transformation

$\Rightarrow$ T is not linear

Recap of 3.5

Ex: $A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & -1 \end{bmatrix}$

Find a basis for:

a) row(A)

nonzero rows of REF/RREF

$R_3 - R_2 \quad \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & -2 \end{bmatrix}$ REF

Basis for row(A) $= \{ [1 \; 1 \; 0 \; 1], [0 \; 1 \; -1 \; 1], [0 \; 0 \; 0 \; -2] \}$

b) col(A)

$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & -2 \end{bmatrix}$ REF

Use columns 1, 2 and 4 of A

Basis for col(A) $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$

c) null(A) $\longleftarrow$ $\{\vec{x} \mid A\vec{x} = \vec{0}\}$

$[A \mid \vec{0}]$

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 & 0 \end{array}\right]$$

$R_1 - R_2$, $\frac{R_3}{-2}$

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

$R_2 - R_3$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & \\ \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \text{ RREF}$$

$x_3 = t$

$x_1 + x_3 = 0 \Rightarrow$ $x_1 = -t$

$x_2 = t$

$x_4 = 0$

$$\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} t$$

Basis for null(A) = $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}$