Final Exam

Mon April 15
1:30 pm
(three hours)
TEC 174 and 175

Test 2

Fri March 1
2.3 - 2.4, 3.1 - 3.3 (6 Questions)
Bring calculator
Bring music / earplugs
Practice Problems on website

3.5 Subspaces and Basis Cont'd

Subspace of $\mathbb{R}^n$
= span of one or more vectors

e.g. line through origin
plane through origin
all of $\mathbb{R}^3$

Basis for a subspace S
= set of direction vectors for S
that contains the minimum # of vectors

Rowspace of A = span of the rows of A

Columnspace of A = " columns "

Example: Let $A = \begin{bmatrix} 2 & 3 & 7 \\ 4 & 7 & 10 \\ 8 & 17 & 8 \end{bmatrix}$. Find a basis for row$(A)$ consisting of rows of A.

Note: This is different from part a) of the previous example, because that answer was not phrased in terms of rows of A.

$$\text{row}(A) = \text{Col}(A^T)$$

Find a basis for $\text{Col}(A^T)$:

$$A^T = \begin{bmatrix} 2 & 4 & 8 \\ 3 & 7 & 17 \\ 7 & 10 & 8 \end{bmatrix}$$

$$\frac{R_1}{2} \begin{bmatrix} 1 & 2 & 4 \\ 3 & 7 & 17 \\ 7 & 10 & 8 \end{bmatrix}$$

$$\begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 7R_1 \end{array} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & -4 & -20 \end{bmatrix}$$

$$\begin{array}{l} \\ \\ R_3 + 4R_2 \end{array} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \text{ REF}$$

Basis for $\text{Col}(A^T) = \{ \text{Columns 1 and 2 of } A^T \}$

$$= \left\{ \begin{bmatrix} 2 \\ 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 10 \end{bmatrix} \right\}$$

or $\{ [2 \ 3 \ 7], [4 \ 7 \ 10] \}$

Example: Let $A = \begin{bmatrix} 1 & 4 & 6 \\ 2 & 8 & 12 \end{bmatrix}$. Find a basis for null($A$).

null (A)
$= \{ \vec{x} \mid A\vec{x} = \vec{0} \}$

Solve $A\vec{x} = \vec{0}$

Each parameter produces a basis vector.

$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 0 \\ 2 & 8 & 12 & 0 \end{array}\right]$$

$R_2 - 2R_1$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 4 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

$x_2 = s$ $\qquad$ $x_3 = t$

$$x_1 + 4x_2 + 6x_3 = 0 \implies x_1 = -4s - 6t$$

$$\vec{x} = \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix} t$$

Basis for null $(A) = \left\{ \begin{bmatrix} -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 0 \\ 1 \end{bmatrix} \right\}$

Example: Find a basis for span($\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\6\end{bmatrix}, \begin{bmatrix}1\\5\\24\end{bmatrix}$).

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 6 \\ 1 & 5 & 24 \end{bmatrix}$$

Find a basis for row(A).

$$\begin{matrix} R_2 - R_1 \\ R_3 - R_1 \end{matrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 6 \\ 0 & 4 & 24 \end{bmatrix}$$

$$R_3 - 4R_2 \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix}_{REF}$$

Basis for row(A) = { nonzero rows of REF/RREF }

$= \{ [1 \ 1 \ 0], [0 \ 1 \ 6] \}$

Definition: Given a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$, the **coordinate vector of $\vec{v}$ with respect to $\mathcal{B}$** is

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{bmatrix} \text{ where } \vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n.$$

Could think of it as the coefficient vector of $\vec{v}$

Example: Find $[\vec{v}]_{\mathcal{B}}$ for $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \}$ and $\vec{v} = \begin{bmatrix} 5 \\ 15 \\ 28 \end{bmatrix}$.

$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{v}$$

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 5 & 1 & 15 \\ 3 & 6 & 4 & 28 \end{array}\right] \end{array}$$

$\vdots$

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right] \end{array}$$

RREF

$c_1 = -2, \quad c_2 = 3, \quad c_3 = 4$

$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 4 \end{bmatrix}$$

Definition: The **dimension** of a subspace S is the number of vectors in a basis for S. It's written $\dim(S)$.

Comment: a) The standard basis for $\mathbb{R}^3 = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$. Therefore $\dim \mathbb{R}^3 = 3$.

b) $\dim \mathbb{R}^n = n$
c) dim(plane through the origin in $\mathbb{R}^n$)= 2
d) dim(line through the origin in $\mathbb{R}^n$)=1

Definition: The **rank** of a matrix is the number of nonzero rows in its REF or RREF.

Comment: For any matrix A: rank(A)=dim(row(A))=dim(col(A)).

Definition: The **nullity** of a matrix A is the number of parameters in the solution to $A\vec{x} = \vec{0}$. In other words, nullity(A)=dim(null(A)).

Example: Let $A = \begin{bmatrix} 1 & 5 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$. Find rank$(A)$ and nullity(A).

$R_3 - R_2$ $\quad \begin{bmatrix} 1 & 5 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ REF

rank(A) = # of nonzero rows in REF/RREF
$= 3$

nullity (A) = # of parameters in solution to $A\vec{x} = \vec{0}$
$= 1$

Fact: For any matrix A: rank(A)+nullity(A)= number of columns in A.

Example: Let's phrase this fact in terms of the columns of A.

$\begin{pmatrix} \text{\# of columns} \\ \text{with pivots} \end{pmatrix} + \begin{pmatrix} \text{\# of columns} \\ \text{without pivots} \end{pmatrix} = \text{\# of columns}$

In Section 2.2 we said:
If a system is consistent then
rank + (# of parameters in solution) = # of variables

The Fundamental Theorem of Invertible Matrices

Let A be an $n \times n$ matrix. The following statements are equivalent:

3.3

a) A is invertible.

b) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.

c) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

d) The RREF of A is I.

e) A is a product of elementary matrices.

f) $\text{rank}(A) = n$.

g) $\text{nullity}(A) = 0$.

h) The columns of A are linearly independent.

i) The span of the columns of A is $\mathbb{R}^n$.

j) The columns of A form a basis for $\mathbb{R}^n$.

k) The rows of A are linearly independent.

l) The span of the rows of A is $\mathbb{R}^n$.

m) The rows of A form a basis for $\mathbb{R}^n$.

Ch4

n) $\det A \neq 0$.

o) 0 is not an eigenvalue of A.

Comment: Consider the Fundamental Theorem of Invertible Matrices. For a given $n \times n$ matrix, the fifteen statements are **all true** or **all false**.

Example: Is $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$ a basis for $\mathbb{R}^3$?

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 5 & 6 \\ 1 & 1 & 4 \end{bmatrix} \qquad \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\det A = 1\begin{vmatrix} 5 & 6 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} 1 & 6 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix}$$

$$= 1(14) - 2(-2) + 3(-4)$$

$$= 6$$

$$\neq 0$$

Yes