Example: Is the following set of vectors a subspace of $\mathbb{R}^3$?

$S = \{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid \; z = x + 1 \}$

span(?)

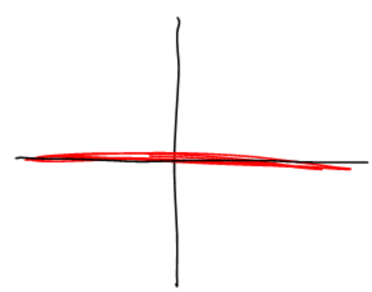
$= \{ \begin{bmatrix} x \\ y \\ x+1 \end{bmatrix} \}$

$\neq \{ x \begin{bmatrix} \# \\ \# \\ \# \end{bmatrix} + y \begin{bmatrix} \# \\ \# \\ \# \end{bmatrix} \}$

No

Example: Is the following set of vectors a subspace of $\mathbb{R}^2$?

$S = \{ \begin{bmatrix} x \\ y \end{bmatrix} \mid \; y = 0 \}$



$= \{ \begin{bmatrix} x \\ 0 \end{bmatrix} \}$

$= \{ x \begin{bmatrix} 1 \\ 0 \end{bmatrix} \}$

$= \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$ YES

Let's define three subspaces associated with a matrix A.

Definition: The **rowspace of** A is the span of the rows of A, written row(A).
The **columnspace of** A is the span of the columns of A, written col(A).
The **nullspace of** A is $\{\vec{x} \mid A\vec{x} = \vec{0}\}$, written null$(A)$.

Example: Let $A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 2 & 1 \end{bmatrix}$.

a) Is $\begin{bmatrix} 6 \\ 10 \end{bmatrix}$ in col(A)?

$$c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

$$\begin{array}{ccc} c_1 & c_2 & c_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 1 & 2 & 1 & 10 \end{array}\right]$$

Solvable $\Rightarrow$ Yes

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 6 \\ 0 & 0 & 1 & 4 \end{array}\right] \text{ REF}$$

b) Is $[1, 2, 5]$ in row(A)?

$$c_1[1 \ \ 2 \ \ 0] + c_2[1 \ \ 2 \ \ 1] = [1 \ \ 2 \ \ 5]$$

$$\begin{array}{cc} c_1 & c_2 \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 0 & 1 & 5 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{array}\right] \text{ REF}$$

Solvable $\Rightarrow$ YES

c) Is $\begin{bmatrix}1\\2\\3\end{bmatrix}$ in null(A)?

$$A\vec{x} = \vec{0}\ ?$$

$$\begin{bmatrix}1 & 2 & 0\\1 & 2 & 1\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} \neq \vec{0}$$

NO

Definition: A set of vectors $\mathcal{B}$ is a **basis for a subspace** S if: span($\mathcal{B}$)=S and $\mathcal{B}$ is linearly independent.

line through origin or plane through origin

Comment: Let's rephrase this. A set $\mathcal{B}$ is a basis for a subspace S if: $\mathcal{B}$ is a set of direction vectors for S containing the minimum number of vectors.

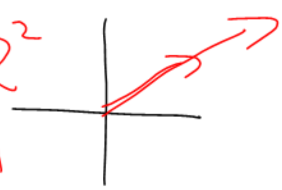
Comment: a) $\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\}$ is a basis for $\mathbb{R}^2$.

b) $\{\begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}\}$ is a basis for $\mathbb{R}^2$.

c) $\{\begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}6\\8\end{bmatrix}\}$ is **not** a basis for $\mathbb{R}^2$.

$\text{span}\left(\begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}6\\8\end{bmatrix}\right) \neq \mathbb{R}^2$

vectors are parallel



d) $\{\begin{bmatrix}3\\4\end{bmatrix}\}$ is **not** a basis for $\mathbb{R}^2$.

$\text{span}\left(\begin{bmatrix}3\\4\end{bmatrix}\right) \neq \mathbb{R}^2$

not enough vectors

e) $\{\begin{bmatrix}3\\4\end{bmatrix}, \begin{bmatrix}5\\6\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\}$ is **not** a basis for $\mathbb{R}^2$.

the vectors are linearly dependent

too many vectors

Example: Let $A = \begin{bmatrix} 2 & 3 & 7 \\ 4 & 7 & 10 \\ 8 & 17 & 8 \end{bmatrix}$. Find a basis for:

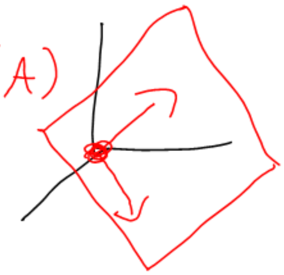
a) row(A)

Use nonzero rows of REF/RREF

$R_2 - 2R_1$
$R_3 - 4R_1$
$\begin{bmatrix} 2 & 3 & 7 \\ 0 & 1 & -4 \\ 0 & 5 & -20 \end{bmatrix}$

$R_3 - 5R_2$
$\begin{bmatrix} 2 & 3 & 7 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$ REF

row(A)



Basis for row(A) $= \{ [2\ 3\ 7], [0\ 1\ -4] \}$

b) col(A)

Use Columns of A corresponding to the pivots in the REF/RREF of A.

$\begin{bmatrix} 2 & 3 & 7 \\ 0 & 1 & -4 \\ 0 & 0 & 0 \end{bmatrix}$ REF

Use Columns 1 and 2 of A

Basis for col(A) $= \left\{ \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 17 \end{bmatrix} \right\}$

col(A)

Comment: In general, performing a row operation changes the columnspace of a matrix. We cannot use the nonzero columns of the REF/RREF to form a basis for col(A).