Test 2

FRI MARCH 1

2.3 - 2.4, 3.1 - 3.3

Fact: To find the LU Factorization of a matrix A:
Transform A to REF using only: (current row)-k(pivot row).
The matrix L has the k-values in the appropriate positions.
The matrix U is the REF.

Fact: The matrix A has an LU Factorization if and only if no row swaps are required to transform A to REF.

Example: Find the LU Factorization of $\begin{bmatrix} 2 & 1 & 1 \\ 4 & 4 & 3 \\ 8 & 10 & 13 \end{bmatrix}$

$$\begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{matrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 6 & 9 \end{bmatrix} \begin{matrix} \\ k=2 \\ k=4 \end{matrix}$$

$$\begin{matrix} \\ \\ R_3 - 3R_2 \end{matrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 6 \end{bmatrix} \begin{matrix} \\ \\ k=3 \end{matrix}$$

REF

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 6 \end{bmatrix}$$

REF

unit lower triangular
k-values in appropriate positions

Example: Let's explore why the method to find the LU Factorization works.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elementary matrix for $R_2 - 2R_1$:

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$(R_2 + 2R_1)$

$$E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

L undoes the operations that turn A into U.

$\Rightarrow$ L turns U into A

$\Rightarrow$ $LU = A$

Example: Find the LU Factorization of A and use it to solve:

$$\begin{bmatrix} 2 & -4 & 0 \\ 3 & -1 & 4 \\ -1 & 2 & 2 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$$

① Find $A = LU$

$$\begin{bmatrix} 2 & -4 & 0 \\ 3 & -1 & 4 \\ -1 & 2 & 2 \end{bmatrix}$$

$$\begin{matrix} R_2 - \frac{3}{2}R_1 \\ R_3 + \frac{1}{2}R_1 \end{matrix} \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} k = \frac{3}{2} \\ k = -\frac{1}{2} \end{matrix}$$

$$\boxed{\begin{aligned} 3 + ?(2) &= 0 \\ ?(2) &= -3 \\ ? &= -\frac{3}{2} \end{aligned}}$$

$$\boxed{\begin{aligned} -1 + ?(2) &= 0 \\ ? &= \frac{1}{2} \end{aligned}}$$

$$R_3 + 0R_2 \begin{bmatrix} 2 & -4 & 0 \\ 0 & 5 & 4 \\ 0 & 0 & 2 \end{bmatrix} k = 0$$

U

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}$$

Example Continued...

2 $\quad L\underbrace{U\vec{x}}_{\vec{y}} = \vec{b}$ $\qquad$ Solve $L\vec{y}=\vec{b}$ for $\vec{y}$

3 $\qquad\qquad\qquad\qquad$ Solve $U\vec{x}=\vec{y}$ for $\vec{x}$

2 $\quad L\vec{y}=\vec{b}$

$$\begin{array}{ccc|c} y_1 & y_2 & y_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 3/2 & 1 & 0 & 0 \\ -1/2 & 0 & 1 & -5 \end{array}\right] \quad \begin{array}{l} y_1 = 2 \\ y_2 = -3 \\ y_3 = -4 \end{array}$$

3 $\quad U\vec{x}=\vec{y}$

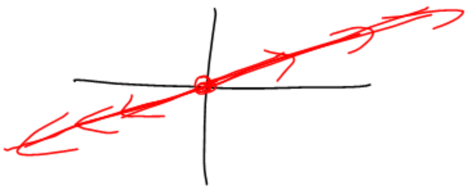
$$\begin{array}{ccc|c} x_1 & x_2 & x_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 2 & -4 & 0 & 2 \\ 0 & 5 & 4 & -3 \\ 0 & 0 & 2 & -4 \end{array}\right] \quad \begin{array}{l} x_1 = 3 \\ x_2 = 1 \\ x_3 = -2 \end{array} \uparrow \qquad \vec{x} = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

3.5 Subspaces and Basis

set of linear combinations

Definition: A **subspace of** $\mathbb{R}^n$ is the span of one or more vectors in $\mathbb{R}^n$.

Comment: a) A line through the origin in $\mathbb{R}^2$ is a subspace of $\mathbb{R}^2$.



$$\vec{x} = t\begin{bmatrix} a \\ b \end{bmatrix}$$

b) A line through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

$$\vec{x} = t\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

c) A plane through the origin in $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$.

$$\vec{x} = s\begin{bmatrix} a \\ b \\ c \end{bmatrix} + t\begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

Example: Is the following set of vectors a subspace of $\mathbb{R}^3$?

$$S = \{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid 3x + 4y + z = 0\}$$

$$= \{\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid z = -3x - 4y\}$$

$$= \{\begin{bmatrix} x \\ y \\ -3x-4y \end{bmatrix}\}$$

$$= \{x\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}\}$$

$$= \mathrm{span}\left(\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -4 \end{bmatrix}\right)$$

YES