Definition: An **elementary matrix** represents a row operation.
To identify which operation, consider how I has been transformed. For example:

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$E = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ represents $2R_1$

$E = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$ represents -$4R_2$

$E = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ represents $R_2 + 3R_1$

$E = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$ represents $R_1 - 5R_2$

$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ represents $R_1 \leftrightarrow R_2$

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ represents $R_2 \leftrightarrow R_3$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ represents $R_2 + 6R_3$

Example: State the row operation that is represented by the elementary matrix. Then find the inverse matrix.

a) $E_1 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$

represents $3R_1$

$\frac{1}{3}R_1$ undoes it

$E_1^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$

b) $E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

represents $R_1 \Leftrightarrow R_2$

$R_1 \Leftrightarrow R_2$ undoes it

$E_2^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

c) $E_3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

represents $R_2 + 2R_1$

$R_2 - 2R_1$ undoes it

$E_3^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

Fact: An elementary matrix acts on the **left** of a matrix. When an elementary matrix is multiplied on the left of A, it performs the associated row operation on A. For example: $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a & 2b \\ c & d \end{bmatrix}$.

Example: Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$. Write A and A^{-1} as a product of elementary matrices.

$\frac{R_1}{2}$ $\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ $\quad E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$ $\quad E_1^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $(2R_1)$

$R_1 - \frac{1}{2}R_2$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad E_2 = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$ $\quad E_2^{-1} = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ $(R_1 + \frac{1}{2}R_2)$

$$\underbrace{E_2 E_1}_{A^{-1}} A = I$$

$$A^{-1} = E_2 E_1$$

$$A = (A^{-1})^{-1}$$
$$= (E_2 E_1)^{-1}$$
$$= E_1^{-1} E_2^{-1}$$

Example: Let $A = \begin{bmatrix} 2 & 4 \\ 1 & 1 \end{bmatrix}$. Write A and A^{-1} as a product of elementary matrices.

$\frac{R_1}{2}$ $\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ $\quad E_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$ $\quad E_1^{-1} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $(2R_1)$

$R_2 - R_1$ $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$ $\quad E_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ $\quad E_2^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ $(R_2 + R_1)$

$\frac{R_2}{(-1)}$ $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $\quad E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $(-R_2)$

$R_1 - 2R_2$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad E_4 = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ $\quad E_4^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ $(R_1 + 2R_2)$

$$\underbrace{E_4 E_3 E_2 E_1}_{A^{-1}} A = I$$

$$A^{-1} = E_4 E_3 E_2 E_1 \checkmark$$

$$\begin{aligned} A &= (A^{-1})^{-1} \\ &= (E_4 E_3 E_2 E_1)^{-1} \\ &= E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} \checkmark \end{aligned}$$

The Fundamental Theorem of Invertible Matrices
Let A be an $n \times n$ matrix. The following statements are equivalent:
a) A is invertible.
b) $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ in $\mathbb{R}^n$.
c) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
d) The RREF of A is I.
e) A is a product of elementary matrices.

Comment: Consider the Fundamental Theorem of Invertible Matrices. For a given $n \times n$ matrix, the five statements are **all true** or **all false**.

Example: Consider the Fundamental Theorem of Invertible Matrices. Which of the five statements are true for A?

a) $A = \begin{bmatrix} 1 & 4 \\ 6 & 9 \end{bmatrix}$

$\det A \neq 0$

$\Rightarrow A$ is invertible

All five statements are true for A.

b) $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

$\det A = 0$

$\Rightarrow A$ is not invertible

None of the five statements are true for A.

3.4 LU Factorization

Definition: An **upper triangular matrix** is a square matrix with zeros below the main diagonal. An example is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$.

Definition: A **lower triangular matrix** is a square matrix with zeros above the main diagonal. An example is $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$.

Definition: A **unit lower triangular matrix** is lower triangular and has ones on the main diagonal. An example is $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 5 & 1 \end{bmatrix}$.

Definition: The LU Factorization of a square matrix A is $A = LU$, where L is a unit lower triangular matrix and U is an upper triangular matrix.

Comment: Here is an LU Factorization:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 4 & 3 \\ 8 & 10 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 6 \end{bmatrix}$$

$A = L\,U$

Example: Solve the system below using the LU Factorization on the previous page.

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & 4 & 3 \\ 8 & 10 & 13 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \\ -8 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

$$L\underbrace{U\vec{x}}_{\vec{y}} = \vec{b}$$

① Solve $L\vec{y} = \vec{b}$ to get $\vec{y}$

② Solve $U\vec{x} = \vec{y}$ to get $\vec{x}$

① $L\vec{y} = \vec{b}$:

$$\begin{array}{ccc} y_1 & y_2 & y_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 2 \\ 4 & 3 & 1 & -8 \end{array}\right]$$

$$y_1 = 1$$

$$2y_1 + y_2 = 2 \Rightarrow 2 + y_2 = 2 \Rightarrow y_2 = 0$$

$$4y_1 + 3y_2 + y_3 = -8 \Rightarrow 4 + 0 + y_3 = -8 \Rightarrow y_3 = -12$$

② $U\vec{x} = \vec{y}$:

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 6 & -12 \end{array}\right]$$

$$\boxed{6x_3 = -12 \Rightarrow x_3 = -2}$$

$$\boxed{2x_2 + x_3 = 0 \Rightarrow x_2 = 1}$$

$$\boxed{2x_1 + x_2 + x_3 = 1 \Rightarrow x_1 = 1}$$

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$