If $A^{-1}A = I$

then $A$ is <u>invertible</u>

and $A^{-1}$ is the <u>inverse of $A$</u>.

"A inverse"

**Definition:** If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the **determinant of** $A$ is $\det A = ad - bc$. (Section 1.4)

**Fact:** If $A$ is a $2 \times 2$ matrix then:

$$A^{-1} = \begin{cases} \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, & \text{if } \det A \neq 0 \\ \text{undefined}, & \text{if } \det A = 0 \end{cases}$$

**Example:** Find $A^{-1}$:

a) $A = \begin{bmatrix} 1 & -4 \\ 7 & 2 \end{bmatrix}$

$\det A = 1(2) - (-4)(7) = 30$

$A^{-1} = \frac{1}{30}\begin{bmatrix} 2 & 4 \\ -7 & 1 \end{bmatrix}$

b) $A = \begin{bmatrix} 3 & -2 \\ -9 & 6 \end{bmatrix}$

$\det A = 18 - 18 = 0$

$A^{-1}$ does not exist

(A is not invertible)

**Fact:** If $A^{-1}$ exists then the system of equations $A\vec{x} = \vec{b}$ has a unique solution: $\vec{x} = A^{-1}\vec{b}$.

**Example:** Let's explore why the above fact is true.

system of equations $A\vec{x} = \vec{b}$

$$\underbrace{A^{-1}A}_{I}\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

**Example:** Use $A^{-1}$ to solve:

$$\begin{aligned} 4x - 5y &= -6 \\ -5x + 6y &= 7 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 4 & -5 \\ -5 & 6 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} -6 \\ 7 \end{bmatrix}}_{\vec{b}}$$

$\det A = -1$

$$A^{-1} = \frac{1}{-1}\begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix} = -\begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix}$$

$$\vec{x} = A^{-1}\vec{b} = -\begin{bmatrix} 6 & 5 \\ 5 & 4 \end{bmatrix}\begin{bmatrix} -6 \\ 7 \end{bmatrix} = -\begin{bmatrix} -1 \\ -2 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$x = 1,\ y = 2$ ✓

**Fact:** To find $A^{-1}$ for an $n \times n$ matrix we form the augmented matrix $[A|I]$. We perform row operations to produce $I$ on the left side. The resulting matrix on the right side will be $A^{-1}$.

**Example:** Find $A^{-1}$ for $A = \begin{bmatrix} 2 & 5 & 1 \\ 1 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$.

$$[A \mid I]$$

$$\left[\begin{array}{ccc|ccc} 2 & 5 & 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$R_1 \Leftrightarrow R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 1 & 0 \\ 2 & 5 & 1 & 1 & 0 & 0 \\ 2 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_2 - 2R_1 \\ R_3 - 2R_1 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 2 & 2 & 0 & 1 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & -2 & -2 & 0 & -2 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 2R_2 \end{array} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & -2 & 5 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & 0 & -8 & 2 & -6 & 1 \end{array}\right]$$

$$\frac{R_3}{-8} \quad \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & -2 & 5 & 0 \\ 0 & 1 & -3 & 1 & -2 & 0 \\ 0 & 0 & 1 & -\frac{2}{8} & \frac{6}{8} & -\frac{1}{8} \end{array}\right]$$

$$\begin{array}{l} R_1 - 8R_3 \\ R_2 + 3R_3 \end{array} \quad \Bigg[\underbrace{\begin{array}{ccc} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}}_{I} \Bigg| \underbrace{\begin{array}{ccc} 0 & -1 & 1 \\ \frac{2}{8} & \frac{2}{8} & -\frac{3}{8} \\ -\frac{2}{8} & \frac{6}{8} & -\frac{1}{8} \end{array}}_{A^{-1}} \Bigg]$$

$[A \mid I] \underset{\text{undoes } A}{\rightsquigarrow} [I \mid A^{-1}]$



**Comment:** By transforming $A$ into $I$ we are "undoing" $A$. The matrix on the right side will be the matrix that "undoes" $A$, that is $A^{-1}$.

**Example:** Find $A^{-1}$ for $A = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 2 & 6 \\ 2 & 3 & 11 \end{bmatrix}$.

$[A \mid I]$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 1 & 2 & 6 & 0 & 1 & 0 \\ 2 & 3 & 11 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 - R_1 \\ R_3 - 2R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & -2 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} R_1 - R_2 \\ \\ R_3 - R_2 \end{array} \left[\begin{array}{ccc|ccc} & & & & & \\ & & & & & \\ 0 & 0 & 0 & & & \end{array}\right]$$

Cannot make $I$ on the left

$\Rightarrow$ $A^{-1}$ does not exist.

**Fact:** Suppose a zero row appears on the left side while reducing $[A|I]$. Then $A^{-1}$ does not exist.

We'll look at three properties of $A^{-1}$.

**Property 1:** If $A^{-1}$ exists then $(A^{-1})^{-1} = A$.

**Property 2:** $(A^T)^{-1} = (A^{-1})^T$ for any matrix $A$.

**Example:** Verify Property 2 for $A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$.

$$(A^T)^{-1} = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}^{-1} = \frac{1}{1}\begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$$

$$(A^{-1})^T = \left(\frac{1}{1}\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}\right)^T = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix}^T = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}$$

**Property 3:** For any matrices $A_1, A_2, \ldots, A_n$ with compatible sizes: $(A_1 A_2 \cdots A_n)^{-1} = A_n^{-1} \cdots A_2^{-1} A_1^{-1}$.

!

**Comment:** In particular this means that $(AB)^{-1} = B^{-1}A^{-1}$.

**Comment:** Let Operation A represent putting on your socks. Let Operation B represent putting on your shoes. To reverse this sequence we have to undo the operations and **reverse the order of operations**. We could express this in matrix terms as $(AB)^{-1} = B^{-1}A^{-1}$.

**Comment:** Consider Property 3 with all $n$ matrices equal to $A$. The statement becomes $(A^n)^{-1} = (A^{-1})^n$. This means we can write $A^{-n}$ without confusion.

**Example:** Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find $A^{-2}$.

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$(A^2)^{-1} = \frac{1}{4} \begin{bmatrix} 22 & -10 \\ -15 & 7 \end{bmatrix}$$

**Example:** Let $A, B$ and $X$ all be invertible $n \times n$ matrices. Solve for $X$ given $(AX)^{-1} = BA$.

$$(AX)^{-1} = BA$$

$$((AX)^{-1})^{-1} = (BA)^{-1}$$

$$AX = (BA)^{-1}$$

Left Multiply by $A^{-1}$

$$\underbrace{A^{-1}A}_{I}X = A^{-1}(BA)^{-1}$$

$$X = A^{-1}(BA)^{-1} \checkmark$$

or $$X = A^{-1}A^{-1}B^{-1} \checkmark$$

or $$X = A^{-2}B^{-1} \checkmark$$

**Definition:** An **elementary matrix** represents a row operation.
To identify which operation, consider how $I$ has been transformed. For example:

$E = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ represents $2R_1$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$E = \begin{bmatrix} 1 & 0 \\ 0 & -4 \end{bmatrix}$ represents -$4R_2$

$E = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$ represents $R_2 + 3R_1$

$E = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$ represents $R_1 - 5R_2$

$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ represents $R_1 \leftrightarrow R_2$

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ represents $R_2 \leftrightarrow R_3$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix}$ represents $R_2 + 6R_3$