3.2 Matrix Algebra

Example: Is $\begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 0 & 1 \\ 6 & 2 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 7 & 2 \end{bmatrix}$?

Let $c_1 \begin{bmatrix} 0 & 1 \\ 6 & 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix}$

$$0c_1 + 0c_2 = 0$$
$$1c_1 + 1c_2 = 1$$
$$6c_1 + 7c_2 = 4$$
$$\cdots$$

$$\begin{array}{cc} c_1 & c_2 \\ \end{array}$$
$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 6 & 7 & 4 \\ 2 & 2 & 2 \end{array}\right]$$

Reorder rows $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 6 & 7 & 4 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \end{array}\right]$

$R_2 - 6R_1$
$R_3 - 2R_1$ $\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ REF

Solvable system

$\Rightarrow$ Yes

Check: $R_1 - R_2$ $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$ RREF $\quad c_1 = 3$, $c_2 = -2$

$$3\begin{bmatrix} 0 & 1 \\ 6 & 2 \end{bmatrix} - 2\begin{bmatrix} 0 & 1 \\ 7 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 4 & 2 \end{bmatrix} \checkmark$$

Example: Find the general form of $\underbrace{\text{span}(\begin{bmatrix}1 & 0\\2 & 0\end{bmatrix}, \begin{bmatrix}3 & 0\\6 & 1\end{bmatrix}, \begin{bmatrix}2 & 1\\4 & 5\end{bmatrix})}_{\text{set of linear combinations}}$.

$$c_1\begin{bmatrix}1 & 0\\2 & 0\end{bmatrix} + c_2\begin{bmatrix}3 & 0\\6 & 1\end{bmatrix} + c_3\begin{bmatrix}2 & 1\\4 & 5\end{bmatrix} = \begin{bmatrix}w & x\\y & z\end{bmatrix}$$

Goal: Find conditions on w, x, y, z

Each zero row of REF will give a condition.

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 3 & 2 & w\\ 0 & 0 & 1 & x\\ 2 & 6 & 4 & y\\ 0 & 1 & 5 & z \end{array}\right] \end{array}$$

$R_3 - 2R_1$ $\left[\begin{array}{ccc|c} 1 & 3 & 2 & w\\ 0 & 0 & 1 & x\\ 0 & 0 & 0 & y-2w\\ 0 & 1 & 5 & z \end{array}\right]$

Reorder rows $\left[\begin{array}{ccc|c} 1 & 3 & 2 & w\\ 0 & 1 & 5 & z\\ 0 & 0 & 1 & x\\ 0 & 0 & 0 & y-2w \end{array}\right]$ REF

Solvable system $\Longrightarrow$ $y - 2w = 0$

$y = 2w$

$\left\{\begin{bmatrix}w & x\\y & z\end{bmatrix} \text{ such that } y = 2w\right\}$ OR $\left\{\begin{bmatrix}w & x\\2w & z\end{bmatrix}\right\}$

Comment: The general form of the span allows us to quickly identify which matrices are in the span. For example, $\begin{bmatrix} 1 & 7 \\ 2 & 30 \end{bmatrix}$ is in the span and $\begin{bmatrix} 1 & 7 \\ 3 & 30 \end{bmatrix}$ is not.

Example: Find the general form of span$\left(\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}\right)$.

$$c_1\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + c_2\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

Get conditions on w, x, y, z.

$$\begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 2 & w \\ 0 & 1 & x \\ 2 & 4 & y \\ 0 & 5 & z \end{array}\right] \end{array}$$

$$R_3 - 2R_1 \quad \left[\begin{array}{cc|c} 1 & 2 & w \\ 0 & 1 & x \\ 0 & 0 & y-2w \\ 0 & 5 & z \end{array}\right]$$

$$R_4 - 5R_2 \quad \left[\begin{array}{cc|c} 1 & 2 & w \\ 0 & 1 & x \\ 0 & 0 & y-2w \\ 0 & 0 & z-5x \end{array}\right] \quad \text{REF}$$

Solvable system

$$\Rightarrow \quad y-2w=0 \quad \text{and} \quad z-5x=0$$
$$y=2w \qquad\qquad z=5x$$

$$\left\{ \begin{bmatrix} w & x \\ y & z \end{bmatrix} \text{ such that } y=2w \text{ and } z=5x \right\} \checkmark$$

OR $\left\{ \begin{bmatrix} w & x \\ 2w & 5x \end{bmatrix} \right\} \checkmark$

Follow-Up: $\begin{bmatrix} 1 & 2 \\ 2 & 10 \end{bmatrix}$ is in the span

$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is not in the span

Example: Are $\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix}$ linearly independent?

$$c_1\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix} + c_2\begin{bmatrix} 3 & 0 \\ 6 & 1 \end{bmatrix} + c_3\begin{bmatrix} 2 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$c_1 = 0$
$c_2 = 0$
$c_3 = 0$

Yes

infinitely-many solutions c_1, c_2, c_3

No

$$\begin{array}{c} \begin{matrix} c_1 & c_2 & c_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 6 & 4 & 0 \\ 0 & 1 & 5 & 0 \end{array}\right] \end{array}$$

$R_3 - 2R_1$
$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 5 & 0 \end{array}\right]$$

Reorder rows
$$\begin{array}{c} \begin{matrix} c_1 & c_2 & c_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$
REF

1 solution
$(c_1 = c_2 = c_3 = 0)$

YES