Example: Balance $NH_3 + O_2 \to N_2 + H_2O$

$$w\, NH_3 + x\, O_2 \to y\, N_2 + z\, H_2O$$

w, x, y, z are the variables

N: $w = 2y \Rightarrow w - 2y = 0$

H: $3w = 2z \Rightarrow 3w - 2z = 0$

O: $2x = z \Rightarrow 2x - z = 0$

$$\begin{array}{cccc} w & x & y & z \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 3 & 0 & 0 & -2 & 0 \\ 0 & 2 & 0 & -1 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 \end{array}\right]$$

$z = t$

$w - \frac{2}{3}z = 0 \Rightarrow w = \frac{2}{3}t$

$x - \frac{1}{2}z = 0 \Rightarrow x = \frac{1}{2}t$

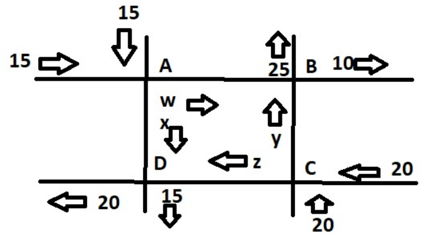
$y - \frac{1}{3}z = 0 \Rightarrow y = \frac{1}{3}t$

$\boxed{\text{Choose smallest positive integer solution: } t = 6}$

$[w, x, y, t] = [4, 3, 2, 6]$

$4NH_3 + 3O_2 \to 2N_2 + 6H_2O$

Example: Consider the following network of one-way streets. The average number of vehicles per hour through intersections A,B,C,D was collected from historical data.
a) Find the flows w, x, y, z.
b) If the solution has a parameter then specify the possible values of the parameter.



Inflow = outflow at each intersection

A: $15+15 = w+x \implies w+x=30$
B: $w+y = 25+10 \implies w+y=35$
C: $20+20 = y+z \implies y+z=40$
D: $x+z = 15+20 \implies x+z=35$

$$\begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 30 \\ 1 & 0 & 1 & 0 & 35 \\ 0 & 0 & 1 & 1 & 40 \\ 0 & 1 & 0 & 1 & 35 \end{array}\right] \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -5 \\ 0 & 1 & 0 & 1 & 35 \\ 0 & 0 & 1 & 1 & 40 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$z = t$ $\quad (t \geq 0)$

$w - z = -5 \implies w = -5 + t \quad (t \geq 5)$
$x + z = 35 \implies x = 35 - t \quad (t \leq 35)$
$y + z = 40 \implies y = 40 - t \quad (t \leq 40)$

$(5 \leq t \leq 35)$ $$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 35 \\ 40 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Example: Find all possible combinations of 15 coins (nickels, dimes or quarters) that total \$ 2.50.

Let x = # of nickels
y = " dimes
z = " quarters

$$x + y + z = 15$$

$$5x + 10y + 25z = 250 \qquad (¢)$$

$$\begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 15 \\ 5 & 10 & 25 & 250 \end{array}\right] \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & -3 & -20 \\ 0 & 1 & 4 & 35 \end{array}\right]$$

$z = t$ $\quad (t \geq 0)$

$x - 3z = -20 \implies x = -20 + 3t \quad (t \geq 7)$

$y + 4z = 35 \implies y = 35 - 4t \quad (t \leq 8)$

Need non-negative integer solutions:

$t = 7$ or $t = 8$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 7 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 8 \end{bmatrix}$$

Chapter 3: Matrices

3.1 Matrix Operations

Definition: The **size** of a matrix is given by (# of rows)×(# of columns).

For example $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is a 2×3 matrix.

Definition: The **entry** of a matrix A is written a_{ij} or $[A]_{ij}$, where i and j are the row index and the column index respectively. For the matrix above $a_{23} = 6$ or $[A]_{23} = 6$.

Definition: A **square** matrix has size $n \times n$.

Definition: An **identity** matrix is square with ones along the main diagonal and zeros elsewhere. It can be written I, or I_n if we want to emphasize its size.

For example $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Definition: A **diagonal** matrix is square and all the entries off the main diagonal are zero.

For example $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ or $D = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$.

Example: Let $A = \begin{bmatrix} 1 & 6 & 1 \\ -2 & -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 6 & 9 \end{bmatrix}$. Find:

a) $A + B$

$$= \begin{bmatrix} 2 & 6 & -2 \\ -1 & 4 & 13 \end{bmatrix}$$

b) $3A$

$$= \begin{bmatrix} 3 & 18 & 3 \\ -6 & -6 & 12 \end{bmatrix}$$

Comment: $A + B$ is undefined if A and B have different sizes.

Definition: The process of multiplying a matrix by a real number is called **scalar multiplication**.

Example: Let $A = \begin{bmatrix} 1 & 6 & 1 \\ -2 & -2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & -3 \\ 1 & 6 & 9 \end{bmatrix}$. Find $A - 3B$.

$$A - 3B = A + (-3B)$$

$$= \begin{bmatrix} 1 & 6 & 1 \\ -2 & -2 & 4 \end{bmatrix} + \begin{bmatrix} -3 & 0 & 9 \\ -3 & -18 & -27 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 6 & 10 \\ -5 & -20 & -23 \end{bmatrix}$$

Definition: The **transpose** of A, written A^T, interchanges the rows and columns of A. The matrix A is **symmetric** if $A^T = A$.

Example: Calculate the transpose and state if the matrix is symmetric.

a) $A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 6 & 3 \\ 4 & 3 & -1 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 6 & 3 \\ 4 & 3 & -1 \end{bmatrix}$$

A is symmetric.

b) $B = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 6 & 1 \end{bmatrix}$

$$B^T = \begin{bmatrix} 1 & 0 \\ 2 & 6 \\ 1 & 1 \end{bmatrix}$$

B is <u>not</u> symmetric.