Example: Are $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\\-1\end{bmatrix}$ and $\begin{bmatrix}0\\0\\2\end{bmatrix}$ linearly independent?

Let $c_1\begin{bmatrix}1\\1\\0\end{bmatrix}+c_2\begin{bmatrix}1\\0\\-1\end{bmatrix}+c_3\begin{bmatrix}0\\0\\2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & -1 & 2 & 0 \end{array}\right] \end{array}$$

$$R_2-R_1 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$\frac{R_2}{-1} \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -1 & 2 & 0 \end{array}\right]$$

$$\begin{array}{l} R_1-R_2 \\ R_3+R_2 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 2 & 0 \end{array}\right]$$

$$\frac{R_3}{2} \quad \begin{array}{c} \begin{array}{ccc} c_1 & c_2 & c_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0 \end{array}\right] \end{array}$$

$c_1 = 0$
$c_2 = 0$
$c_3 = 0$

Yes. Vectors are linearly independent.

Fact: A set of more than n vectors in $\mathbb{R}^n$ is linearly dependent. For example three vectors in $\mathbb{R}^2$ are guaranteed to be linearly dependent.

Example: Let's explore why this fact is true.

Let $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_m\vec{v}_m = \vec{0}$

$$n\left\{\begin{array}{c} \begin{array}{cccc} c_1 & c_2 & \cdots & c_m \end{array} \\ \underbrace{\left[\begin{array}{cccc|c} & & & & 0 \\ & & & & 0 \\ & & & & \vdots \\ & & & & 0 \end{array}\right]}_{m > n} \end{array}\right.$$

System is solvable because $c_1 = c_2 = \dots = c_m = 0$ solves the system.

$$\left[\begin{array}{ccccc|c} 1 & & & & & 0 \\ & 1 & & & & 0 \\ & & 1 & & & \vdots \\ & & & \ddots & 1 & 0 \end{array}\right] \quad \text{RREF}$$

Solution has at least 1 parameter.

$\Rightarrow$ There are infinitely-many solutions $c_1, c_2, \dots, c_m$.

$\Rightarrow$ The vectors are linearly dependent.

Example: Find a linear dependence relationship (linear dependency) involving $\begin{bmatrix}1\\6\end{bmatrix}$, $\begin{bmatrix}2\\6\end{bmatrix}$ and $\begin{bmatrix}4\\30\end{bmatrix}$. Start by letting $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

$$\begin{array}{c} \begin{matrix} c_1 & c_2 & c_3 & \end{matrix} \\ \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 6 & 6 & 30 & 0 \end{array}\right] \end{array}$$

$$R_2 - 6R_1 \quad \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & -6 & 6 & 0 \end{array}\right]$$

$$\frac{R_2}{-6} \quad \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\begin{array}{c} \begin{matrix} c_1 & c_2 & c_3 & \end{matrix} \\ R_1 - 2R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & 6 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \end{array} \quad \text{RREF}$$

$$c_3 = t$$

$$c_1 + 6c_3 = 0 \implies c_1 = -6t$$

$$c_2 - c_3 = 0 \implies c_2 = t$$

Let t be any nonzero real # : $t = 1$

$$c_1 = -6, \quad c_2 = 1, \quad c_3 = 1$$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$$

$$-6\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}$$

Example: Find a linear dependence relationship (linear dependency) involving $\begin{bmatrix}1\\6\end{bmatrix}$, $\begin{bmatrix}2\\6\end{bmatrix}$ and $\begin{bmatrix}4\\30\end{bmatrix}$. Start by putting the vectors into the rows of a matrix.

$$\begin{array}{c} \vec{v}_1 \\ \vec{v}_2 \\ \vec{v}_3 \end{array} \begin{bmatrix} 1 & 6 \\ 2 & 6 \\ 4 & 30 \end{bmatrix}$$

$$\begin{array}{c} \vec{v}_1 \\ \vec{v}_2 - 2\vec{v}_1 \\ \vec{v}_3 - 4\vec{v}_1 \end{array} \begin{bmatrix} 1 & 6 \\ 0 & -6 \\ 0 & 6 \end{bmatrix}$$

$$\begin{array}{c} \\ \\ (\vec{v}_3 - 4\vec{v}_1) + (\vec{v}_2 - 2\vec{v}_1) \end{array} \begin{bmatrix} & \\ & \\ 0 & 0 \end{bmatrix}$$

Any zero row gives a dependency.

$$\vec{v}_3 - 4\vec{v}_1 + \vec{v}_2 - 2\vec{v}_1 = \vec{0}$$

$$-6\vec{v}_1 + \vec{v}_2 + \vec{v}_3 = \vec{0}$$

Comment: Compare the methods used in the last two examples. The first method gives the general solution, while the second method gives one particular solution.

Comment: Preview of Section 3.5:
We'll consider objects like lines or planes through the origin, and find a set of direction vectors containing the minimum number of vectors. This discussion will require knowledge of span and linear independence.

2.4 Applications of Linear Systems

Example: Find the parabola $y = ax^2 + bx + c$ that passes through $(1, 12), (-1, 18)$ and $(2, 30)$.

$ax^2 + bx + c = y$

a, b, c are the variables

$(x, y) = (1, 12)$: $\quad a + b + c = 12$

$(-1, 18)$: $\quad a - b + c = 18$

$(2, 30)$: $\quad 4a + 2b + c = 30$

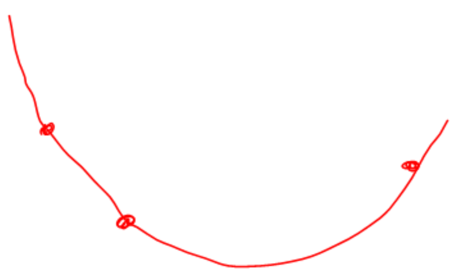
$$\begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 12 \\ 1 & -1 & 1 & 18 \\ 4 & 2 & 1 & 30 \end{array}\right] \end{array}$$

$$\rightsquigarrow \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 8 \end{array}\right] \end{array}$$

$a = 7, \quad b = -3, \quad c = 8$

$y = ax^2 + bx + c$

$y = 7x^2 - 3x + 8$



Example: Balance $NH_3 + O_2 \to N_2 + H_2O$

$$w\, NH_3 + x\, O_2 \to y\, N_2 + z\, H_2O$$

w, x, y, z are the variables

$$\begin{aligned}
N:&\quad w = 2y &\Longrightarrow&\quad w - 2y = 0\\
H:&\quad 3w = 2z &\Longrightarrow&\quad 3w - 2z = 0\\
O:&\quad 2x = z &\Longrightarrow&\quad 2x - z = 0
\end{aligned}$$

$$\begin{array}{cccc} w & x & y & z \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0\\ 3 & 0 & 0 & -2 & 0\\ 0 & 2 & 0 & -1 & 0 \end{array}\right]$$

To Be Continued