2.3 Span and Linear Independence

Example: Is $\begin{bmatrix} 8 \\ -10 \end{bmatrix}$ a linear combination of $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$?

Let $c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 8 \\ -10 \end{bmatrix}$

$$\begin{cases} -c_1 + 2c_2 = 8 \\ 2c_1 - 3c_2 = -10 \end{cases}$$

$$\begin{array}{ccc} c_1 & c_2 & \\ \end{array}$$
$$\left[\begin{array}{cc|c} -1 & 2 & 8 \\ 2 & -3 & -10 \end{array}\right]$$

$$\frac{R_1}{-1} \quad \left[\begin{array}{cc|c} 1 & -2 & -8 \\ 2 & -3 & -10 \end{array}\right]$$

$$R_2 - 2R_1 \quad \left[\begin{array}{cc|c} 1 & -2 & -8 \\ 0 & 1 & 6 \end{array}\right] \quad \text{REF}$$

System is consistent.

Yes

To check: $R_1 + 2R_2$
$$\begin{array}{ccc} c_1 & c_2 & \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 6 \end{array}\right] \quad \text{RREF}$$

$c_1 = 4, \quad c_2 = 6$

$$4 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + 6 \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 8 \\ -10 \end{bmatrix} \checkmark$$

Example: Is $\vec{w} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ a linear combination of $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$?

Let $\underbrace{c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}}_{\text{linear combination of } \vec{u} \text{ and } \vec{v}} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

$$\begin{array}{c} \begin{array}{cc} c_1 & c_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 1 & 0 & 2 \end{array}\right] \end{array}$$

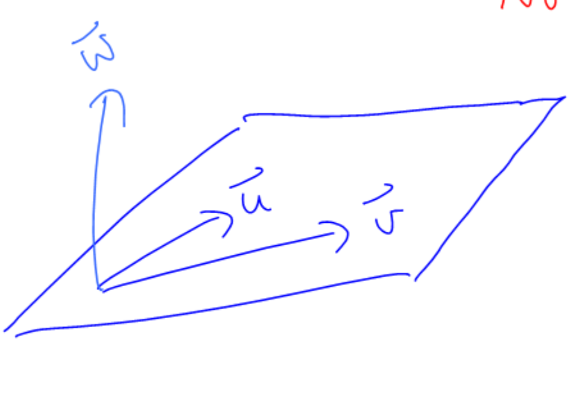
$$R_3 - R_1 \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{array}\right]$$

$$\frac{R_2}{3} \quad \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & \frac{1}{3} \\ 0 & -1 & 1 \end{array}\right]$$

$$R_3 + R_2 \quad \left[\begin{array}{cc|c} & & \\ & & \\ 0 & 0 & \frac{4}{3} \end{array}\right] \quad \text{REF}$$

No solution

No



Fact: The vector $\vec{b}$ is a linear combination of the columns of matrix A if and only if the system $\left[A|\ \vec{b}\right]$ is consistent.

Definition: The **span** of $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ is the set of all linear combinations of $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$.

Comment: a) span($\vec{a}, \vec{b}$)= $\{\vec{0},\ 3\vec{a},\ -7\vec{b},\ 2\vec{a}+5\vec{b}, \ldots\}$

b) span($\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$)= $\{c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_n\vec{u}_n\}$
where $c_1, c_2, \ldots, c_n$ are any real numbers.

Fact: The zero vector $\vec{0}$ is in span$(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n)$ because $0\vec{u}_1 + 0\vec{u}_2 + \cdots + 0\vec{u}_n = \vec{0}$.

Example: Describe each span geometrically:

a) span($\begin{bmatrix}1\\1\end{bmatrix}$, $\begin{bmatrix}-3\\-3\end{bmatrix}$)

$= \{ c_1 \begin{bmatrix}1\\1\end{bmatrix} + c_2 \begin{bmatrix}-3\\-3\end{bmatrix} \}$

$= \{ t \begin{bmatrix}1\\1\end{bmatrix} \}$

Line through the origin in $\mathbb{R}^2$ with $\vec{d} = \begin{bmatrix}1\\1\end{bmatrix}$

b) span($\begin{bmatrix}1\\1\end{bmatrix}$, $\begin{bmatrix}1\\2\end{bmatrix}$)

All of $\mathbb{R}^2$ (the xy-plane)

c) span($\begin{bmatrix}1\\0\\-1\end{bmatrix}$, $\begin{bmatrix}-4\\0\\4\end{bmatrix}$)

$= \{ c_1 \begin{bmatrix}1\\0\\-1\end{bmatrix} + c_2 \begin{bmatrix}-4\\0\\4\end{bmatrix} \}$

$= \{ t \begin{bmatrix}1\\0\\-1\end{bmatrix} \}$

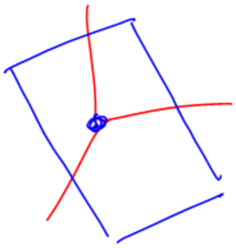
Line through the origin in $\mathbb{R}^3$ with $\vec{d} = \begin{bmatrix}1\\0\\-1\end{bmatrix}$

d) span($\begin{bmatrix}1\\6\\0\end{bmatrix}$, $\begin{bmatrix}2\\3\\0\end{bmatrix}$)

Plane through the origin in $\mathbb{R}^3$ with direction vectors $\bar{u} = \begin{bmatrix}1\\6\\0\end{bmatrix}$ and $\bar{v} = \begin{bmatrix}2\\3\\0\end{bmatrix}$

Example: Find an equation for span($\begin{bmatrix}1\\0\\1\end{bmatrix}$, $\begin{bmatrix}1\\3\\6\end{bmatrix}$). Give your answer in any form.

Plane through origin



NORMAL FORM

$\vec{n} = \vec{u} \times \vec{v}$

$= [-3, -5, 3]$

1 0 1 1 0
1 3 6 1 3

$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$

$\begin{bmatrix}-3\\-5\\3\end{bmatrix} \cdot \begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-3\\-5\\3\end{bmatrix} \cdot \begin{bmatrix}0\\0\\0\end{bmatrix}$ ✓

VECTOR FORM

$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$

$\vec{x} = \begin{bmatrix}0\\0\\0\end{bmatrix} + s\begin{bmatrix}1\\0\\1\end{bmatrix} + t\begin{bmatrix}1\\3\\6\end{bmatrix}$ ✓

Example: a) Show that span($\begin{bmatrix}1\\3\end{bmatrix}$, $\begin{bmatrix}2\\1\end{bmatrix}$)$= \mathbb{R}^2$. (algebra)

Let $c_1\begin{bmatrix}1\\3\end{bmatrix} + c_2\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$

Show the system is solvable for c_1 and c_2.

$$\begin{array}{cc} c_1 & c_2 \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 2 & a \\ 3 & 1 & b \end{array}\right]$$

$$R_2 - 3R_1 \quad \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & -5 & b-3a \end{array}\right] \text{ REF}$$

System is solvable for c_1 and c_2. ✓

b) Write $\begin{bmatrix}a\\b\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\3\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$.

$$\frac{R_2}{-5} \quad \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 1 & \frac{b-3a}{-5} \end{array}\right]$$

$$a + \frac{2}{5}(b-3a) = a + \frac{2b}{5} - \frac{6a}{5} = -\frac{a}{5} + \frac{2b}{5}$$

$$R_1 - 2R_2 \quad \left[\begin{array}{cc|c} 1 & 0 & \frac{-a}{5} + \frac{2b}{5} \\ 0 & 1 & \frac{3a}{5} - \frac{b}{5} \end{array}\right] \text{ RREF}$$

$$c_1 = \frac{-a}{5} + \frac{2b}{5}$$

$$c_2 = \frac{3a}{5} - \frac{b}{5}$$

$$\left(-\frac{a}{5} + \frac{2b}{5}\right)\begin{bmatrix}1\\3\end{bmatrix} + \left(\frac{3a}{5} - \frac{b}{5}\right)\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$$

Comment: To decide if a system is consistent, reduce it to REF.
To solve a system, reduce it to RREF.

Definition: Given $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$, consider solutions to $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$. If the only solution is $c_1 = c_2 = \ldots = c_n = 0$ then the set of vectors is **linearly independent**. If there are solutions other than $c_1 = c_2 = \ldots = c_n = 0$ then the set of vectors is **linearly dependent**.

Let $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_n\vec{v}_n = \vec{0}$

$c_1 = c_2 = \ldots = c_n = 0$ is the only solution

Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are <u>linearly independent</u>

$c_1 = c_2 = \ldots = c_n = 0$ and other solutions

Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are <u>linearly dependent</u>.

Comment: The two sentences below mean the same thing:
Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly independent.
The set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly independent.

Comment: The two sentences below mean the same thing:
Vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are linearly dependent.
The set $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is linearly dependent.

Comment: a) $\{\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}, \begin{bmatrix}2\\7\end{bmatrix}\}$ is linearly dependent.

$$3\begin{bmatrix}0\\1\end{bmatrix} + 1\begin{bmatrix}2\\4\end{bmatrix} - 1\begin{bmatrix}2\\7\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \checkmark$$

b) $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}3\\3\end{bmatrix}$ are linearly dependent.

$$0\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}1\\1\end{bmatrix} - 1\begin{bmatrix}3\\3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Coefficients are not all zero.

c) $\begin{bmatrix}0\\0\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$ are linearly dependent.

$$1\begin{bmatrix}0\\0\end{bmatrix} + 0\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

Coefficients are not all zero