Example: Solve:

$$\begin{aligned} 2x - 3y &= 8 \\ -4x + 6y &= 20 \end{aligned}$$

$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ -4 & 6 & 20 \end{array}\right]$$

Get a 1

$$\frac{R_1}{2} \quad \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 4 \\ -4 & 6 & 20 \end{array}\right]$$

Get 0's in rest of Column 1

$$R_2 + 4R_1 \quad \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 4 \\ 0 & 0 & 36 \end{array}\right]$$

Current row − # (pivot row)

$0x + 0y = 36$ impossible

The system has no solution.

Fact: A system has no solution if the following type of row appears while performing row operations:
[all zeros | nonzero]

Example: Solve:

$$\begin{aligned} 2x - 3y &= 8 \\ -4x + 6y &= -16 \end{aligned}$$

$$\begin{array}{ccc} x & y & \# \\ \end{array}$$
$$\left[\begin{array}{cc|c} 2 & -3 & 8 \\ -4 & 6 & -16 \end{array}\right]$$

Get a 1

$$\frac{R_1}{2} \quad \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 4 \\ -4 & 6 & -16 \end{array}\right]$$

Get 0's in rest of Column 1

$$\begin{array}{ccc} x & y & \# \\ \end{array}$$
$$R_2 + 4R_1 \quad \left[\begin{array}{cc|c} 1 & -\frac{3}{2} & 4 \\ 0 & 0 & 0 \end{array}\right]$$

Current Row − # (Pivot Row)

Column ~~for~~ y has no pivot.

y is a <u>free variable</u>.

$y = t$ (t is a <u>parameter</u>)

$1x - \frac{3}{2}y = 4 \Rightarrow x = 4 + \frac{3}{2}y \Rightarrow x = 4 + \frac{3}{2}t$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix}$$

System has infinitely-many solutions.

Example: Solve:

$$\begin{aligned} x &= 5 \\ 2x+3y &= 4 \\ 3x+4y &= 7 \end{aligned}$$

$$\left[\begin{array}{cc|c} 1 & 0 & 5 \\ 2 & 3 & 4 \\ 3 & 4 & 7 \end{array}\right]$$

Get 0's in rest of Column 1

$$\begin{array}{l} R_2-2R_1 \\ R_3-3R_1 \end{array} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 3 & -6 \\ 0 & 4 & -8 \end{array}\right]$$

$$\frac{R_2}{3} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 4 & -8 \end{array}\right]$$

$$\begin{array}{ccc} x & y & \# \end{array}$$

$$R_3-4R_2 \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right] \Leftarrow 0x+0y=0$$

No info

$$1x+0y=5 \Rightarrow x=5$$

$$y=-2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \quad \text{1 unique solution}$$

Definition: **Back-substitution** is the process of solving a sytem from the bottom equation upwards.

Example: Solve by back-substitution:

$$\begin{aligned} 4x + \ y + \ \ z &= 15 \\ 3y + 5z &= 29 \\ 2z &= 8 \end{aligned}$$

$$2z = 8 \implies z = 4$$

$$3y + 5z = 29 \implies 3y + 20 = 29 \implies y = 3$$

$$4x + y + z = 15 \implies 4x + 3 + 4 = 15 \implies x = 2$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Comment: Most systems can't be solved by back-substitution.

2.2 Solving Systems

Definition: A matrix is in **row-echelon form** (REF) if:
any zero rows are at the bottom AND
the leading nonzero entries of each row move down and right

Comment: The following matrices are in REF:

$$\begin{bmatrix} 6 & 0 & -1 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & 3 & -1 \\ 0 & 4 & 7 \\ 0 & 0 & 0 \end{bmatrix}$$

Definition: An augmented matrix is in REF if the coefficient matrix is in REF.

Comment: The following matrices are in REF:

$$\left[\begin{array}{ccc|c} 6 & 0 & -1 & 1 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 3 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 0 & 4 & 7 & 0 \\ 0 & 0 & 0 & 9 \end{array}\right]$$

Definition: One method of solving a system is **Gaussian Elimination**. The augmented matrix is transformed to REF using elementary row operations. The system is then solved by back-substitution.

Example: Solve by Gaussian Elimination:

$$\begin{aligned} x + 2y + z &= 6 \\ 2x + 2y &= 8 \\ 3y + z &= 8 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 2 & 2 & 0 & 8 \\ 0 & 3 & 1 & 8 \end{array}\right]$$

$$R_2 - 2R_1 \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & -2 & -2 & -4 \\ 0 & 3 & 1 & 8 \end{array}\right]$$

$$\frac{R_2}{-2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 3 & 1 & 8 \end{array}\right]$$

$$R_3 - 3R_2 \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -2 & 2 \end{array}\right] \quad \text{REF}$$

$$-2z = 2 \implies z = -1$$

$$y + z = 2 \implies y - 1 = 2 \implies y = 3$$

$$x + 2y + z = 6 \implies x + 6 - 1 = 6 \implies x = 1$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

Definition: A matrix is in **reduced row-echelon form** (RREF) if:
the matrix is in REF,
the leading nonzero entry in each row is 1, AND
these leading ones have zeros everywhere else in their columns

Comment: The following matrices are in RREF:

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comment: The following matrix is in REF but not RREF:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

Definition: An augmented matrix is in RREF if the coefficient matrix is in RREF.

Comment: The following matrices are in RREF:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 1 & 5 & 0 & 9 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 9 \end{array}\right]$$

Definition: Another method of solving a system is **Gauss-Jordan Elimination**. The augmented matrix is transformed to RREF using elementary row operations. This is typically faster than Gaussian Elimination.

Example: Solve by Gauss-Jordan Elimination:

$$\begin{aligned} x + 2y + 3z &= 7 \\ 3x + 3y + 3z &= 15 \\ 5x + 7y + 9z &= 29 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 3 & 3 & 3 & 15 \\ 5 & 7 & 9 & 29 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 - 3R_1 \\ R_3 - 5R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & -3 & -6 & -6 \\ 0 & -3 & -6 & -6 \end{array}\right]$$

$$\begin{array}{l} \\ \frac{R_2}{-3} \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 7 \\ 0 & 1 & 2 & 2 \\ 0 & -3 & -6 & -6 \end{array}\right]$$

$$\begin{array}{l} R_1 - 2R_2 \\ \\ R_3 + 3R_2 \end{array} \begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 3 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \quad \text{RREF}$$

Circle the leading non-zero entry in each <u>row</u>.
Any <u>column</u> without a circle gets a parameter.

$$z = t$$

$$1x + 0y - 1z = 3 \Rightarrow x = 3 + z \Rightarrow x = 3 + t$$
$$y + 2z = 2 \Rightarrow y = 2 - 2z \Rightarrow y = 2 - 2t$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Example: Solve by Gauss-Jordan Elimination:

$$\begin{aligned} x + \ y - \ \ 6z &= 17 \\ 2x + 2y - \ \ 8z &= 22 \\ 3x + 3y - 14z &= 39 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & -6 & 17 \\ 2 & 2 & -8 & 22 \\ 3 & 3 & -14 & 39 \end{array}\right]$$

$$\begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -6 & 17 \\ 0 & 0 & 4 & -12 \\ 0 & 0 & 4 & -12 \end{array}\right]$$

$$\frac{R_2}{4} \left[\begin{array}{ccc|c} 1 & 1 & -6 & 17 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 4 & -12 \end{array}\right]$$

$$\begin{array}{l} R_1 + 6R_2 \\ \\ R_3 - 4R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

$$y = k$$

$$x + y = -1 \Rightarrow x = -1 - y \Rightarrow x = -1 - k$$

$$z = -3$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -3 \end{bmatrix} + k \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$