1.4 The Cross Product

The cross product $\vec{u} \times \vec{v}$ is defined for $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$.

Example: Let $\vec{u} = [1, 2, 1]$ and $\vec{v} = [3, -1, 4]$. Calculate $\vec{u} \times \vec{v}$.

$$\begin{array}{ccccc} 1 & 2 & 1 & 1 & 2 \\ 3 & -1 & 4 & 3 & -1 \end{array}$$

$$\vec{u} \times \vec{v} = [2(4) - 1(-1), 1(3) - 1(4), 1(-1) - 2(3)]$$
$$= [9, -1, -7]$$

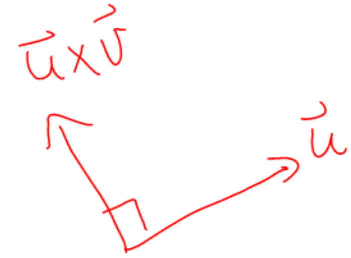
Example: Let $\vec{u} = [1, 2, 1]$ and $\vec{v} = [3, -1, 4]$. Calculate:
a) $\vec{v} \times \vec{u}$

$$\begin{array}{ccccc} 3 & -1 & 4 & 3 & -1 \\ 1 & 2 & 1 & 1 & 2 \end{array}$$

$$\vec{v} \times \vec{u} = [-9, 1, 7]$$

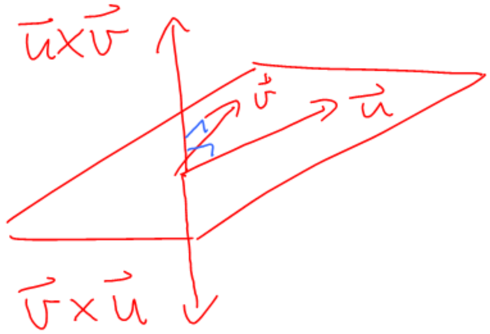
b) $(\vec{u} \times \vec{v}) \cdot \vec{u}$

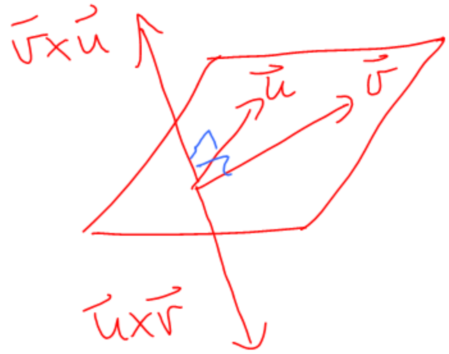
$$= [9, -1, -7] \cdot [1, 2, 1]$$
$$= 9(1) + (-1)(2) + (-7)(1)$$
$$= 0$$



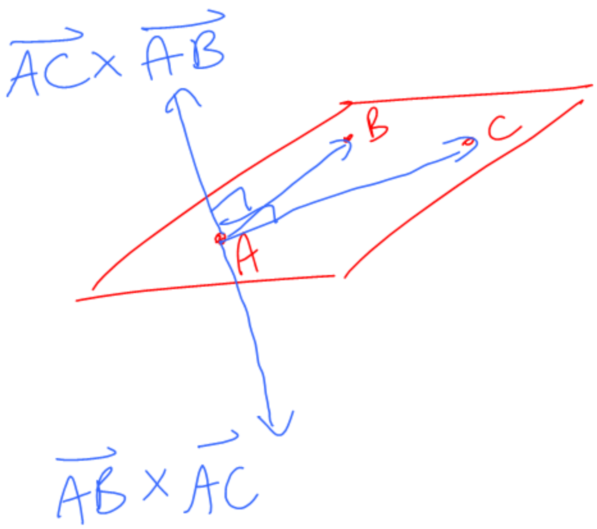
Fact: Let $\vec{u}$ and $\vec{v}$ be in $\mathbb{R}^3$. Then:
$\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$ AND
$\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$

Fact: The vector $\vec{u} \times \vec{v}$ is a normal for the plane containing $\vec{u}$ and $\vec{v}$. The direction of $\vec{u} \times \vec{v}$ is determined by the Right Hand Rule.





Example: Find the general form of the plane through $A = (1, 3, 6)$, $B = (2, 1, 4)$ and $C = (1, -1, 5)$.



$$\vec{AB} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$$

$$\vec{AC} = \begin{bmatrix} 0 \\ -4 \\ -1 \end{bmatrix}$$

$$\vec{AB} \times \vec{AC} = [-6, 1, -4]$$

$$\boxed{\begin{matrix} 1 & -2 & -2 & 1 & -2 \\ 0 & -4 & -1 & 0 & -4 \end{matrix}}$$

$$\vec{n} = \begin{bmatrix} -6 \\ 1 \\ -4 \end{bmatrix}$$

Normal form $\quad \begin{bmatrix} -6 \\ 1 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -6 \\ 1 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix}$

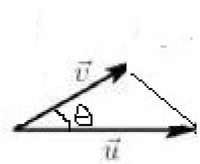
General form $\quad -6x + y - 4z = -27$ ✓

or $\quad 6x - y + 4z = 27$ ✓

Comment: Recall that $\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta$ for $\vec{u}, \vec{v}$ in $\mathbb{R}^n$.

Fact: If $\vec{u}$ and $\vec{v}$ are in $\mathbb{R}^3$ then $||\vec{u} \times \vec{v}|| = ||\vec{u}||\ ||\vec{v}|| \sin\theta$.

Example: Let $\vec{u}$ and $\vec{v}$ be in $\mathbb{R}^3$. Consider the triangle below.
Show that the area of the triangle is $\frac{1}{2}||\vec{u} \times \vec{v}||$

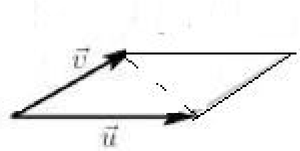


$$\sin\theta = \frac{h}{||\vec{v}||}$$

$$h = ||\vec{v}|| \sin\theta$$

$$A(\text{triangle}) = \frac{1}{2}\, b\, h$$
$$= \frac{1}{2} ||\vec{u}||\ ||\vec{v}|| \sin\theta$$
$$= \frac{1}{2} ||\vec{u} \times \vec{v}||$$

Fact: Let $\vec{u}$ and $\vec{v}$ be in $\mathbb{R}^3$. Consider the parallelogram below, which can be divided into two triangles with equal area. Then:
Area(triangle)= $\frac{1}{2}||\vec{u} \times \vec{v}||$ AND
Area(parallelogram)= $||\vec{u} \times \vec{v}||$



Example: Find the area of the triangle determined by $\vec{u} = [1, 4, 5]$ and $\vec{v} = [2, 3, 6]$.

$$\begin{array}{ccccc} 1 & 4 & 5 & 1 & 4 \\ 2 & 3 & 6 & 2 & 3 \end{array}$$

$$\vec{u} \times \vec{v} = [9, 4, -5]$$

$$\|\vec{u} \times \vec{v}\| = \sqrt{9^2 + 4^2 + (-5)^2}$$

$$= \sqrt{122}$$

$$A(\text{triangle}) = \frac{\|\vec{u} \times \vec{v}\|}{2}$$

$$= \frac{\sqrt{122}}{2}$$