No Formula Sheet for Math 251

Test 1

Fri Feb 2

1.1–1.4, 2.1–2.2

1.3 Lines and Planes

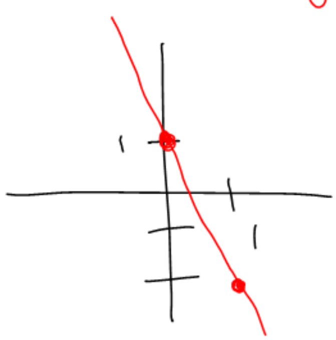
Part 1. Lines in $\mathbb{R}^2$

Definition: The **general form** of a line in $\mathbb{R}^2$ is $ax + by = c$

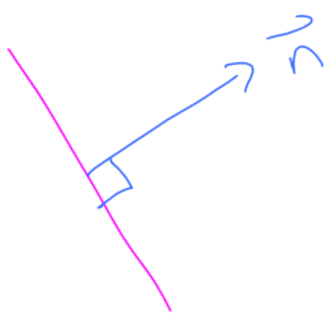
Example: Consider the line $3x + y = 1$. Find two points on the line and graph the line.

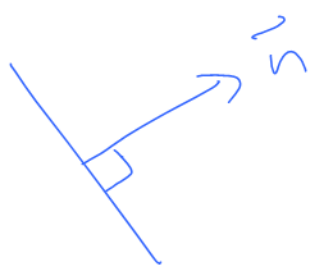
Set $x = 0$: $y = 1$ $\qquad P(0, 1)$

Set $x = 1$: $3 + y = 1$
$y = -2$ $\qquad Q(1, -2)$



Definition: A **normal vector** is orthogonal to a given line. It is written $\vec{n}$. Its components are the coefficients from the general form.





$3x + y = 1$ $\qquad \vec{n} = [3, 1]$

Definition: The **normal form** of a line in $\mathbb{R}^2$ is $\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$ where $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\vec{p}$ is the vectorization of any point on the line.

Example: Describe the line $3x + y = 1$ in normal form. Show that expanding normal form gives general form.

$\vec{n} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$

$P = (0, 1)$ $\quad \vec{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

normal form

$\vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}$

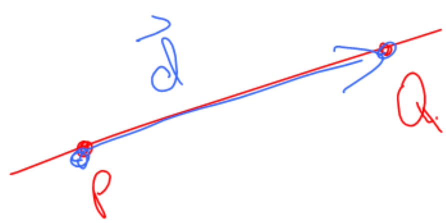
$\begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

normal form

$3x + y = 1$

general form

Definition: A **direction vector** for a line is $\vec{d} = \overrightarrow{PQ}$, where P and Q are any two points on the line.



Definition: The **vector form** for a line in $\mathbb{R}^2$ is $\vec{x} = \vec{p} + t\vec{d}$, where t represents any real number.

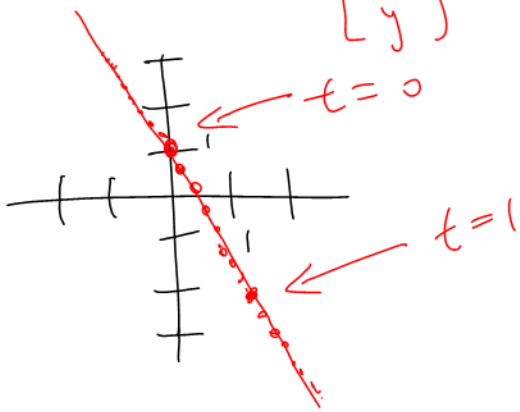
Example: Describe the line $3x + y = 1$ in vector form. Show that as t varies, the line is traced out.

$P = (0, 1) \qquad Q = (1, -2)$

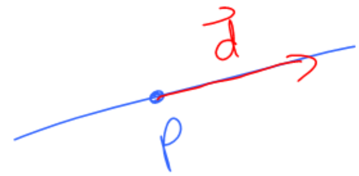
direction vector $\vec{d} = \vec{PQ} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ Think $Q - P$

vector form $\vec{x} = \vec{p} + t\vec{d}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$



$\vec{x} = \vec{p} + t\vec{d}$



Definition: The **parametric form** for a line in $\mathbb{R}^2$ is:

$$\begin{cases} x = a + bt \\ y = c + dt \end{cases}$$

Example: Describe the line $3x + y = 1$ in parametric form.

Vector Form $\vec{x} = \vec{p} + t\vec{d}$

Vector Form $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Expand: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} t \\ -3t \end{bmatrix}$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 1 - 3t \end{bmatrix}$$

Parametric Form $\begin{cases} x = t \\ y = 1 - 3t \end{cases}$

Comment: A given line can be described in a specific form in multiple ways, for example $3x + y = 1$ and $6x + 2y = 2$ are general forms for the same line.

Example: Summarize the four forms of a line in $\mathbb{R}^2$

general form $3x + y = 1$

normal form $\begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

vector form $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

parametric form $\begin{cases} x = t \\ y = 1 - 3t \end{cases}$

Part 2. Lines in $\mathbb{R}^3$

Example: Consider the line through $P = (2, 1, 12)$ and $Q = (0, -3, 6)$. Describe the line in both vector and parametric form.

Vector form $\qquad \vec{x} = \vec{p} + t\vec{d}$

$$\vec{d} = \vec{PQ} = \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$$ Think $Q - P$

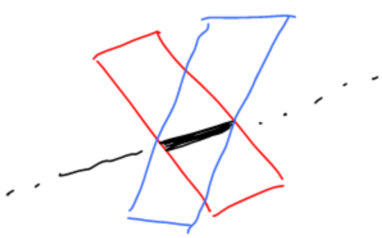
Vector form $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 12 \end{bmatrix} + t \begin{bmatrix} -2 \\ -4 \\ -6 \end{bmatrix}$$

Parametric form $$\begin{cases} x = 2 - 2t \\ y = 1 - 4t \\ z = 12 - 6t \end{cases}$$

Definition: A **plane** is an infinite flat surface.

Fact: $ax + by + cz = d$ is the general form for a plane in $\mathbb{R}^3$.

Comment: General form for a line in $\mathbb{R}^3$ is inconvenient so we will omit it. It would consist of two equations, describing the intersection of two planes.

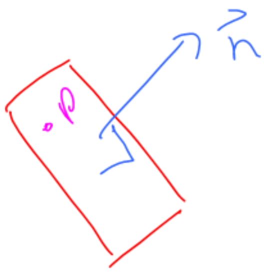


Line in $\mathbb{R}^3$: $$\begin{cases} 2x + 4y + 5z = 9 \\ 3x + 2y + z = 7 \end{cases}$$

Comment: Similarly we omit normal form for a line in $\mathbb{R}^3$.

Part 3. Planes in $\mathbb{R}^3$

Example: Consider the plane through $P = (1, -1, 3)$ with normal $\begin{bmatrix}1\\1\\2\end{bmatrix}$. Describe the plane in both normal and general form.

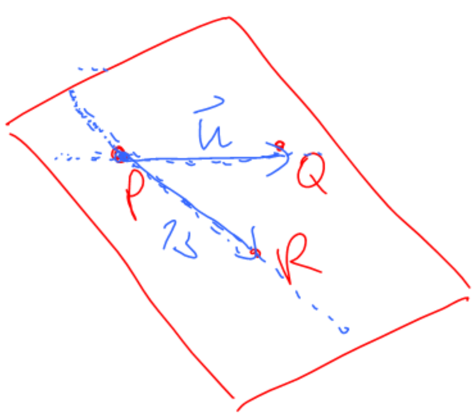


normal form $\vec{n}\cdot\vec{x} = \vec{n}\cdot\vec{p}$

normal form $\begin{bmatrix}1\\1\\2\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\1\\2\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\3\end{bmatrix}$

general form $x + y + 2z = 6$

Definition: The **vector form** for a plane in $\mathbb{R}^3$ is $\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$ where:
$\vec{u}$ and $\vec{v}$ are nonparallel direction vectors
s and t represent any real numbers



$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$