Review Problems

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$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 5 & 1 & a \\ 2 & 1 & 3 & b \\ 7 & -19 & 13 & c \end{array}\right] \end{array}$$

$$\begin{array}{r} \\ R_2-2R_1 \\ R_3-7R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 5 & 1 & a \\ 0 & -9 & 1 & b-2a \\ 0 & -54 & 6 & c-7a \end{array}\right]$$

$$\begin{array}{r} \\ \\ R_3-6R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 5 & 1 & a \\ 0 & -9 & 1 & b-2a \\ 0 & 0 & 0 & 5a-6b+c \end{array}\right] \quad \text{REF}$$

$$c-7a-6(b-2a)$$
$$= c-7a-6b+12a$$
$$= 5a-6b+c$$

Any zero row of REF produces a condition.

$$\boxed{5a-6b+c=0}$$

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$$\left[\begin{array}{cc|c} 1 & k & 1 \\ k & 1 & 1 \end{array}\right]$$

$$R_2-kR_1 \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1-k^2 & 1-k \end{array}\right]$$

$1-k^2 \neq 0$

$$\frac{R_2}{1-k^2} \quad \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 1 & k & 1 \\ 0 & 1 & \frac{1-k}{1-k^2} \end{array}\right] \end{array}$$

1 solution

$1-k^2=0$

$1=k^2$

$k^2=1$

$k=\pm 1$

$$\boxed{k=1} \qquad \boxed{k=-1}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 0 & 2 \end{array}\right]$$

∞-many solutions $\qquad$ no solution

$$\begin{cases} 0 \text{ solutions} & \text{if } k=-1 \\ 1 \text{ solution} & \text{if } 1-k^2 \neq 0 \\ \infty\text{-many solutions} & \text{if } k=1 \end{cases}$$

(12)

$$\text{span} = \left\{ c_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Let $c_1 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$

$$\begin{array}{ccc} c_1 & c_2 & c_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & w \\ 1 & 1 & 0 & x \\ 1 & 0 & 0 & y \\ 1 & 0 & 1 & z \end{array}\right]$$

Get conditions on w, x, y, z.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & w \\ 0 & 0 & 0 & x-w \\ 0 & -1 & 0 & y-w \\ 0 & -1 & 1 & z-w \end{array}\right]$$

$\left(\frac{R_4}{-1}\right)$ then $R_2 \leftrightarrow R_4$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & w \\ 0 & 1 & -1 & w-z \\ 0 & -1 & 0 & y-w \\ 0 & 0 & 0 & x-w \end{array}\right]$

$$R_3 + R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 0 & w \\ 0 & 1 & -1 & w-z \\ 0 & 0 & -1 & y-z \\ 0 & 0 & 0 & x-w \end{array}\right] \quad \text{REF}$$

If system is consistent then $x - w = 0$.

$$\Rightarrow x = w$$

The span is $\left\{ \begin{bmatrix} w & x \\ y & z \end{bmatrix} \text{ such that } x = w \right\}$

$$= \left\{ \begin{bmatrix} w & w \\ y & z \end{bmatrix} \right\}$$

(14) $$E_3 E_2 E_1 A = I$$

Recall: elementary matrices act on the <u>left</u> of A.

$$A^{-1} = E_3 E_2 E_1$$

$$A = (A^{-1})^{-1}$$

$$= E_1^{-1} E_2^{-1} E_3^{-1} \quad \text{(order reverses)}$$

$$= \underset{R_2 \leftrightarrow R_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}} \underset{2R_3}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}} \underset{R_1 \to R_1 + 3R_3}{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\boxed{I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

16 a) nonzero rows of REF / RREF

$$\{ [1 \ 0 \ -1 \ -2], [0 \ 1 \ 2 \ 3] \}$$

b) $\text{row}(A) = \text{col}(A^T)$

Use the pivots as a guide for which columns to select.

$$\begin{bmatrix} 1 & & 1 & \\ & & & \end{bmatrix}$$

Use columns 1 and 3 of A^T

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

c)

$$\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & \end{bmatrix}$$

Use columns 1 and 2 of A

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \\ 4 \end{bmatrix} \right\}$$

d) $\text{null}(A) = \{ \vec{x} \mid A\vec{x} = \vec{0} \}$

$$[A \mid \vec{0}]$$

$$[RREF \mid \vec{0}]$$

$$\begin{array}{cccc|c} w & x & y & z & \\ 1 & 0 & -1 & -2 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

$$y = s \quad z = t$$

$$w - y - 2z = 0 \implies w = s + 2t$$

$$x = -2s - 3t$$

$$\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} t$$

Basis for null(A) = $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$

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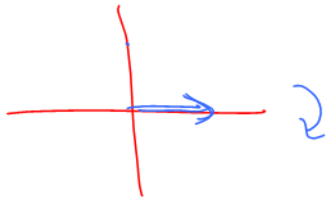
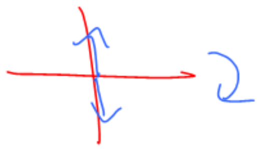
$$[S] = \begin{bmatrix} \underset{S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} & \underset{S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)}{\begin{pmatrix} 0 \\ -1 \end{pmatrix}} \end{bmatrix}$$





$$[T] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta = \frac{\pi}{3}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$$

$$S\left(T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)\right) = [S][T]\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 - \sqrt{3} \\ \sqrt{3} + 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 - \sqrt{3} \\ -\sqrt{3} - 1 \end{bmatrix}$$

19 $c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

$$\begin{array}{cc} c_1 & c_2 \end{array}$$
$$\left[\begin{array}{cc|c} 2 & 0 & 6 \\ 1 & 1 & 8 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 5 \end{array}\right]$$

$c_1 = 3 \qquad c_2 = 5$

$$T\left(\begin{bmatrix} 6 \\ 8 \end{bmatrix}\right) = T\left(3\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

T linear

$\Rightarrow T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$
and
$T(c\vec{x}) = cT(\vec{x})$

$$= T\left(3\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) + T\left(5\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= 3\,T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) + 5\,T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= 3\begin{bmatrix} 5 \\ 6 \end{bmatrix} + 5\begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 30 \\ 38 \end{bmatrix}$$