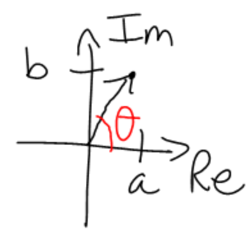
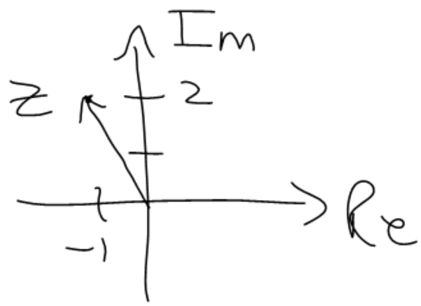
$z = a + bi$



Definition: The **length** of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.
The **principal argument** of $z = a + bi$ is the angle $\theta = \tan^{-1}(\frac{b}{a})$ $(+\pi?)$
We decide whether to add π or not based on the graph of z.

Example: Let $z = -1 + 2i$. Graph z then calculate $|z|$ and θ.

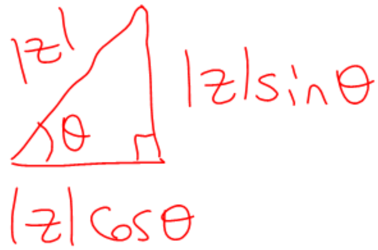


$$|z| = \sqrt{(-1)^2 + 2^2}$$
$$= \sqrt{5}$$

$$\theta = \tan^{-1}\frac{b}{a} \quad (+\pi?)$$
$$= \tan^{-1}\left(\frac{2}{-1}\right) + \pi$$
$$\approx 2.03 \text{ rads}$$

Add π when the real part of z is < 0.

Example: Show that $z = |z|[\cos\theta + i\sin\theta]$.



$$z = |z|\cos\theta + i|z|\sin\theta$$
$$= |z|[\cos\theta + i\sin\theta]$$

Definition: The **rectangular form** of a complex number is $z = a + bi$.
The **polar form** of a complex number is $z = |z|[\cos\theta + i\sin\theta]$.

Example: Express $z = -1 + 8i$ in polar form.

$$|z| = \sqrt{(-1)^2 + 8^2}$$
$$= \sqrt{65}$$

$$\theta = \tan^{-1}\left(\frac{8}{-1}\right) \quad (+\pi?)$$
$$= \tan^{-1}(-8) + \pi \quad \checkmark$$
$$= -\tan^{-1}8 + \pi \quad \checkmark$$
$$= \pi - \tan^{-1}8 \quad \checkmark$$

$$z = |z|[\cos\theta + i\sin\theta]$$
$$= \sqrt{65}\left[\cos(\pi - \tan^{-1}8) + i\sin(\pi - \tan^{-1}8)\right]$$

Fact: Multiplication and Division in Polar Form

Let $z_1 = |z_1|[\cos\theta_1 + i\sin\theta_1]$ and $z_2 = |z_2|[\cos\theta_2 + i\sin\theta_2]$.

Then $z_1 z_2 = |z_1|\,|z_2|[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$ and

$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$.

Comment: When multiplying in polar form: multiply the lengths and add the angles. When dividing in polar form: divide the lengths and subtract the angles.

Example: Let $z_1 = 9 + 3\sqrt{3}i$ and $z_2 = 4\sqrt{3} - 12i$.
Find $\frac{z_1}{z_2}$ by converting to polar form.

Convert z_1 to polar:

$$|z_1| = \sqrt{9^2 + (3\sqrt{3})^2} = \sqrt{108} = 6\sqrt{3}$$

$$\theta_1 = \tan^{-1}\left(\frac{3\sqrt{3}}{9}\right) \cancel{(+\pi?)}$$

$$= \tan^{-1}\frac{1}{\sqrt{3}}$$

$$= \frac{\pi}{6}$$

$$z_1 = |z_1|[\cos\theta_1 + i\sin\theta_1]$$

$$z_1 = 6\sqrt{3}\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$$

$$|z_2| = \sqrt{(4\sqrt{3})^2 + (-12)^2} = \sqrt{192} = 8\sqrt{3}$$

$$\theta_2 = \tan^{-1}\left(\frac{-12}{4\sqrt{3}}\right) \cancel{(+\pi?)}$$

$$= \tan^{-1}(-\sqrt{3})$$

$$= -\frac{\pi}{3}$$

$$z_2 = 8\sqrt{3}\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$$

Example Continued...

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$$

$$= \frac{6\sqrt{3}}{8\sqrt{3}}\left[\cos\left(\frac{\pi}{6} + \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right)\right]$$

$$= \frac{3}{4}\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$$

polar form

$$= \frac{3}{4}\left[0 + i\right]$$

$$= \frac{3}{4}i$$

rectangular form

Fact: De Moivre's Formula
Let n be a positive integer.
If $z = |z|[\cos\theta + i\sin\theta]$ then $z^n = |z|^n[\cos(n\theta) + i\sin(n\theta)]$.

$\boxed{\text{lengths multiply, angles add}}$

Example: Find $(1-i)^{21}$.

Let $z = 1 - i$

$$|z| = \sqrt{1^2 + (-1)^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{-1}{1}\right) \cancel{(+\pi?)}$$

$$= \tan^{-1}(-1)$$

$$= -\frac{\pi}{4}$$

$$z^{21} = \sqrt{2}^{21}\left[\cos\left(-\frac{21\pi}{4}\right) + i\sin\left(-\frac{21\pi}{4}\right)\right]$$

polar

$$= \sqrt{2}^{20}\sqrt{2}\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right]$$

$$\boxed{-\frac{21\pi}{4} + \frac{24\pi}{4} = \frac{3\pi}{4}}$$

$$= 1024\sqrt{2}\left[-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right]$$

$$= -1024 + 1024i$$

rectangular