Assignment

Due Mon April 8

Exam

Mon April 15
1:30 pm
TEC 174/175

Office Hours

Mon 15th 12-1pm

Exam Breakdown

Three Hours
15 Questions, with Parts

Ch	% of Marks on Exam
1	18
2	21
3	23
4	13
5	11
7.3 and Complex	14

Example: Find the best-fit parabola through:

x	y
1	1
2	-2
3	3
4	4

Let $y = a_0 + a_1 x + a_2 x^2$

(a_0, a_1, a_2 are the unknowns)

$$1(a_0) + x(a_1) + x^2(a_2) = y$$

$$\begin{array}{ccc} 1 & x & x^2 \end{array}$$
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{array}{c} y \\ \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix} \end{array}$$

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} = \begin{bmatrix} 4 & 10 & 30 \\ 10 & 30 & 100 \\ 30 & 100 & 354 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{20} \begin{bmatrix} 155 & -135 & 25 \\ -135 & 129 & -25 \\ 25 & -25 & 5 \end{bmatrix}$$

$[A \mid I] \rightsquigarrow [I \mid A^{-1}]$
or adjoint method
or MATLAB

$$\vec{x}^* = \frac{1}{20} \begin{bmatrix} & & \\ & & \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 3 \\ 4 \end{bmatrix}}$$

$$= \frac{1}{20} \begin{bmatrix} & & \\ & & \end{bmatrix} \begin{bmatrix} 6 \\ 22 \\ 84 \end{bmatrix}$$

$$= \frac{1}{20} \begin{bmatrix} 60 \\ -72 \\ 20 \end{bmatrix} \begin{array}{l} \Leftarrow a_0 = 3 \\ \Leftarrow a_1 = -18/5 \\ \Leftarrow a_2 = 1 \end{array}$$

$$y = 3 - \frac{18}{5}x + x^2$$

Octave Online
M = [4 10 30;
... 30 100 354]
format rat
inv(M)

Example: Find the best-fit curve $P = Ce^{kt}$ through:

t	P
0	5
1	8
3	12

$$\ln P = \ln(Ce^{kt})$$

$$\ln P = \ln C + \ln e^{kt}$$

$$\ln P = \ln C + kt$$

$\ln C$ and k are the unknowns

$$1(\ln C) + tk = \ln P$$

$$\begin{array}{cc} 1 & t \\ \left[\begin{array}{c} 1 \\ 1 \\ 1 \end{array}\right. & \left.\begin{array}{c} 0 \\ 1 \\ 3 \end{array}\right] \end{array} \begin{bmatrix} \ln C \\ k \end{bmatrix} = \begin{array}{c} \ln P \\ \begin{bmatrix} \ln 5 \\ \ln 8 \\ \ln 12 \end{bmatrix} \end{array}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 4 & 10 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{14} \begin{bmatrix} 10 & -4 \\ -4 & 3 \end{bmatrix}$$

$$\vec{x}^* = (A^T A)^{-1} A^T \vec{b}$$

$$= \frac{1}{14} \begin{bmatrix} 10 & -4 \\ -4 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} \ln 5 \\ \ln 8 \\ \ln 12 \end{bmatrix}}$$

Example Continued...

$$= \frac{1}{14}\begin{bmatrix} 10 & 6 & -2 \\ -4 & -1 & 5 \end{bmatrix}\begin{bmatrix} \ln 5 \\ \ln 8 \\ \ln 12 \end{bmatrix}$$

$$= \frac{1}{14}\begin{bmatrix} 10\ln 5 + 6\ln 8 - 2\ln 12 \\ -4\ln 5 - \ln 8 + 5\ln 12 \end{bmatrix}$$

$$\approx \begin{bmatrix} 1.69 \\ 0.279 \end{bmatrix} \begin{matrix} \leftarrow \ln C \\ \leftarrow k \end{matrix}$$

$$k \approx 0.3$$

$$C = e^{\ln C} \approx 5$$

$$P = Ce^{kt}$$

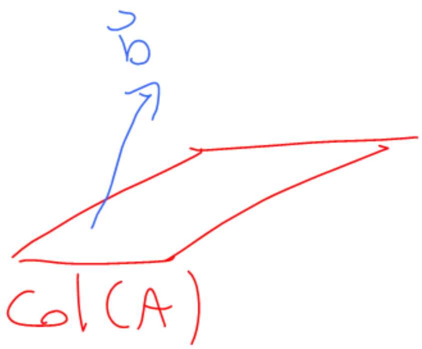
$$P = 5e^{0.3t}$$

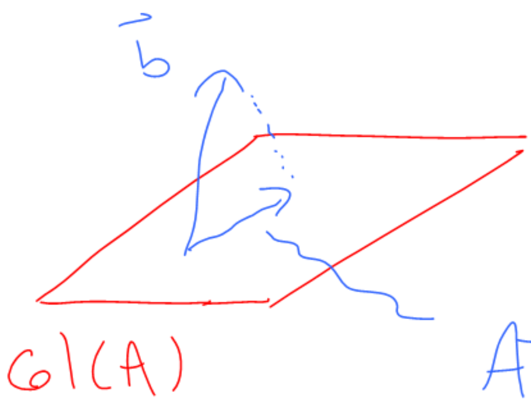
solvable

Recall that $A\vec{x} = \vec{b}$ is consistent if and only if $\vec{b}$ is in col(A).
This follows from Sections 2.3 and 3.5.

Recall that null(A^T)=[col(A)]$^\perp$. We saw this in Section 5.2.

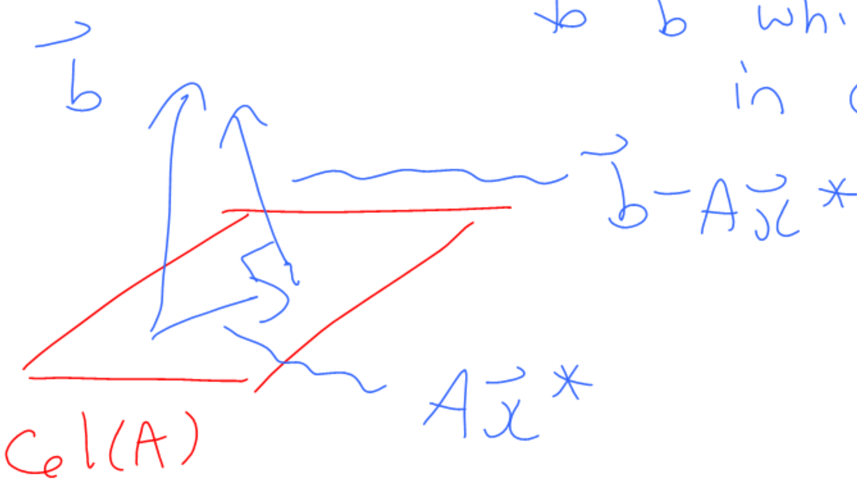
Example: Derive the formula for $\vec{x}^*$ by considering an inconsistent system $A\vec{x} = \vec{b}$.





$A\vec{x}^* = \text{proj}_{\text{col}(A)}\vec{b}$

(closest we can get ~~to~~ $\vec{b}$ while remaining in col(A))



Example Continued...

$$\vec{b} - A\vec{x}^* \text{ is in } [\mathrm{Col}(A)]^\perp$$

$$\vec{b} - A\vec{x}^* \text{ is in } \mathrm{null}(A^T)$$

$$A^T(\vec{b} - A\vec{x}^*) = \vec{0}$$

$$A^T\vec{b} - A^TA\vec{x}^* = \vec{0}$$

$$A^T\vec{b} = A^TA\vec{x}^*$$

$$\cancel{(A^TA)^{-1}A^TA}\,\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$$

$$\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$$

Complex Numbers

Definition: Let i be the **imaginary number** such that $i^2 = -1$.
If a and b are real numbers then $z = a + bi$ is a **complex number**.

Comment: The symbol i is sometimes written j. You may feel free to use either notation.

Example: Let $z_1 = -2 + 6i$ and $z_2 = 4 + 5i$. Calculate:

a) $-7z_1$

$= 14 - 42i$

b) $z_1 + z_2$

$= 2 + 11i$

c) $z_1 - z_2$

$= -6 + i$

d) $z_1 z_2$

$= (-2+6i)(4+5i)$

$= -8 - 10i + 24i + \cancel{30i^2}$ -30

$= -38 + 14i$

Definition: The **complex conjugate** of $z = a + bi$ is $\overline{z} = a - bi$.

Example: Let $z = a + bi$. Show that $z\overline{z} = a^2 + b^2$.

real #

$$z\overline{z} = (a+bi)(a-bi)$$
$$= a^2 - abi + abi \cancel{- b^2 i^2} + b^2$$
$$= a^2 + b^2 \leftarrow \text{real \#}$$

Example: Let $z_1 = 4 + 9i$ and $z_2 = -3 + 5i$. Calculate:

a) $\frac{1}{z_1}$

$$= \frac{1}{(4+9i)} \frac{(4-9i)}{(4-9i)}$$

Multiply top and bottom by conjugate of denominator

$$= \frac{4-9i}{97} \leftarrow a^2+b^2$$

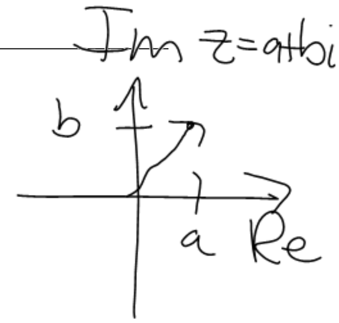
$$= \frac{4}{97} - \frac{9}{97} i$$

b) $\frac{z_1}{z_2}$

$$= \frac{(4+9i)}{(-3+5i)} \frac{(-3-5i)}{(-3-5i)}$$

$$= \frac{-12 - 20i - 27i \cancel{- 45 i^2} + 45}{34} \leftarrow a^2+b^2$$

$$= \frac{33 - 47i}{34} \quad \text{or} \quad \frac{33}{34} - \frac{47}{34} i$$



Definition: The **length** of $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.
The **principal argument** of $z = a + bi$ is the angle $\theta = \tan^{-1}(\frac{b}{a})$ $(+\pi?)$
We decide whether to add π or not based on the graph of z.

Example: Let $z = -1 + 2i$. Graph z then calculate $|z|$ and θ.