Example: Approximating the eigenvalues of A. **This example will not be tested**.

Consider the following procedure:
Find $A = Q_0 R_0$.
Let $A_1 = R_0 Q_0$ then find $A_1 = Q_1 R_1$.
Let $A_2 = R_1 Q_1$ then find $A_2 = Q_2 R_2$ etc.
Each matrix A_k has the same eigenvalues as A.
As $k \to \infty$, A_k becomes upper triangular.

Suppose we start with matrix A and produce $A_4 = \begin{bmatrix} 1.98 & 2.52 \\ 0.03 & 7.01 \end{bmatrix}$.

a) Does A_4 have the same eigenvalues as A?

Yes

b) Is A_4 approximately upper triangular?

$0.03 \approx 0$ Yes

c) Estimate the eigenvalues of A.

$\lambda \approx 1.98, 7.01$

5.4 Orthogonal Diagonalization

Recall that if Q is orthogonal then $Q^{-1} = Q^T$.

Definition: An $n \times n$ matrix A is **orthogonally diagonalizable** if there exist an orthogonal matrix Q and a diagonal matrix D so that $Q^T AQ = D$.

(Compare with Section 4.4 $P^{-1}AP = D$)

Fact: Let A be an $n \times n$ matrix. The matrix A is orthogonally diagonalizable if and only if A is symmetric.

$A^T = A$

Example: Is A orthogonally diagonalizable?

a) $A = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix}$

$A^T = A$ ✓

Yes

b) $A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

$A^T \neq A$

No

Example: The matrix $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$ has eigenvalues $\lambda = 4$ and $\lambda = 7$.

Find Q that orthogonally diagonalizes A.

To find Q: find an orthonormal basis for each eigenspace.

$\lambda = 7$: $[A - 7I \mid \vec{0}]$

$$\begin{array}{c} \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \\ \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

$x_3 = t$

$x_1 - x_3 = 0 \implies x_1 = t$

$x_2 - x_3 = 0 \implies x_2 = t$

$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} t$ A basis for $E_7 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

An orthonormal basis for $E_7 = \left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$\lambda = 4$: $[A - 4I \mid \vec{0}]$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$$

Example Continued...

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \text{RREF}$$

$x_2 = s \qquad x_3 = t$

$$x_1 + x_2 + x_3 = 0 \implies x_1 = -s - t$$

$$\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} t \qquad \text{A basis for } E_4 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Gram-Schmidt

Partial Basis $X = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \text{proj}_X \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$2\vec{v}_2 = 2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Orthogonal Basis for $E_4 = \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$

Orthonormal " "

$$= \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$$

$$Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$$

To check:

$$Q^T A Q = \begin{bmatrix} 7 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

We're going to recap the outer product expansion of AB from Section 3.1.

Example: Find $\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}5 & 6\\ 7 & 8\end{bmatrix}$ using the outer product expansion.

$$= \begin{bmatrix}1\\ 3\end{bmatrix}\begin{bmatrix}5 & 6\end{bmatrix} + \begin{bmatrix}2\\ 4\end{bmatrix}\begin{bmatrix}7 & 8\end{bmatrix}$$

$$= \begin{bmatrix}5 & 6\\ 15 & 18\end{bmatrix} + \begin{bmatrix}14 & 16\\ 28 & 32\end{bmatrix}$$

$$= \begin{bmatrix}19 & 22\\ 43 & 50\end{bmatrix} \checkmark$$

Example: Find $\begin{bmatrix}-1 & 9\\ 2 & 3\end{bmatrix}\begin{bmatrix}4 & 3\\ 2 & 1\end{bmatrix}$ using the outer product expansion.

$$= \begin{bmatrix}-1\\ 2\end{bmatrix}\begin{bmatrix}4 & 3\end{bmatrix} + \begin{bmatrix}9\\ 3\end{bmatrix}\begin{bmatrix}2 & 1\end{bmatrix}$$

$$= \begin{bmatrix}-4 & -3\\ 8 & 6\end{bmatrix} + \begin{bmatrix}18 & 9\\ 6 & 3\end{bmatrix}$$

$$= \begin{bmatrix}14 & 6\\ 14 & 9\end{bmatrix}$$

Definition: Let A be a symmetric $n \times n$ matrix.
Let $\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n$ be orthonormal eigenvectors written as columns.
Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the corresponding eigenvalues.
The **spectral decomposition of** A is:
$A = \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \lambda_2 \vec{q}_2 \vec{q}_2^{\,T} + \ldots + \lambda_n \vec{q}_n \vec{q}_n^{\,T}$

Example: Find a 3×3 matrix A with eigenvalues $\lambda = 2$ and $\lambda = 3$ so that:
$E_2 = \text{span}(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix})$ and $E_3 = \text{span}(\begin{bmatrix}1\\1\\-2\end{bmatrix})$.

orthogonal eigenvectors ✓

Orthonormal eigenvectors: $\vec{q}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$, $\vec{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\\0\end{bmatrix}$, $\vec{q}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix}1\\1\\-2\end{bmatrix}$

$\lambda_1 = 2 \qquad \lambda_2 = 2 \qquad \lambda_3 = 3$

$$A = \lambda_1 \vec{q}_1 \vec{q}_1^{\,T} + \lambda_2 \vec{q}_2 \vec{q}_2^{\,T} + \lambda_3 \vec{q}_3 \vec{q}_3^{\,T}$$

$$= \cancel{2\frac{1}{\sqrt{3}}\frac{1}{\sqrt{3}}}\ \frac{2}{3} \begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\end{bmatrix} + \cancel{\frac{2}{\sqrt{2}\cdot\sqrt{2}}}\ 1 \begin{bmatrix}1\\-1\\0\end{bmatrix}\begin{bmatrix}1 & -1 & 0\end{bmatrix}$$

$$+ \cancel{\frac{3}{\sqrt{6}\cdot\sqrt{6}}}\ \frac{1}{2} \begin{bmatrix}1\\1\\-2\end{bmatrix}\begin{bmatrix}1 & 1 & -2\end{bmatrix}$$

$$= \frac{4}{6}\begin{bmatrix}1 & 1 & 1\\1 & 1 & 1\\1 & 1 & 1\end{bmatrix} + \frac{6}{6}\begin{bmatrix}1 & -1 & 0\\-1 & 1 & 0\\0 & 0 & 0\end{bmatrix}$$

$$+ \frac{3}{6}\begin{bmatrix}1 & 1 & -2\\1 & 1 & -2\\-2 & -2 & 4\end{bmatrix}$$

Example Continued...

$$= \frac{1}{6}\begin{bmatrix} 13 & 1 & -2 \\ 1 & 13 & -2 \\ -2 & -2 & 16 \end{bmatrix} \leftarrow \text{symmetric}$$

Check:

$$A\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \checkmark$$

$$A\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \checkmark$$

$$A\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \checkmark$$