Example: Approximating the eigenvalues of A. This example will not be tested.

Consider the following procedure: Find  $A = Q_0 R_0$ . Let  $A_1 = R_0 Q_0$  then find  $A_1 = Q_1 R_1$ . Let  $A_2 = R_1 Q_1$  then find  $A_2 = Q_2 R_2$  etc. Each matrix  $A_k$  has the same eigenvalues as A. As  $k \to \infty$ ,  $A_k$  becomes upper triangular.

Suppose we start with matrix A and produce  $A_4 = \begin{bmatrix} 1.98 & 2.52 \\ 0.03 & 7.01 \end{bmatrix}$ . a) Does  $A_4$  have the same eigenvalues as A?



b) Is  $A_4$  approximately upper triangular?

c) Estimate the eigenvalues of A.

$$\lambda \approx 1.98$$
, 7.0(

## 5.4 Orthogonal Diagonalization

Recall that if Q is orthogonal then  $Q^{-1} = Q^T$ .

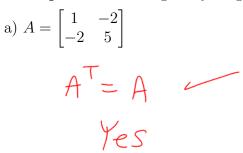
**Definition:** An  $n \times n$  matrix A is **orthogonally diagonalizable** if there exist an orthogonal matrix Q and a diagonal matrix D so that  $Q^T A Q = D$ .

(Gapare with Section 4.4  $P^{-1}AP = D$ )

**Fact:** Let A be an  $n \times n$  matrix. The matrix A is orthogonally diagonalizable if and only if A is symmetric.

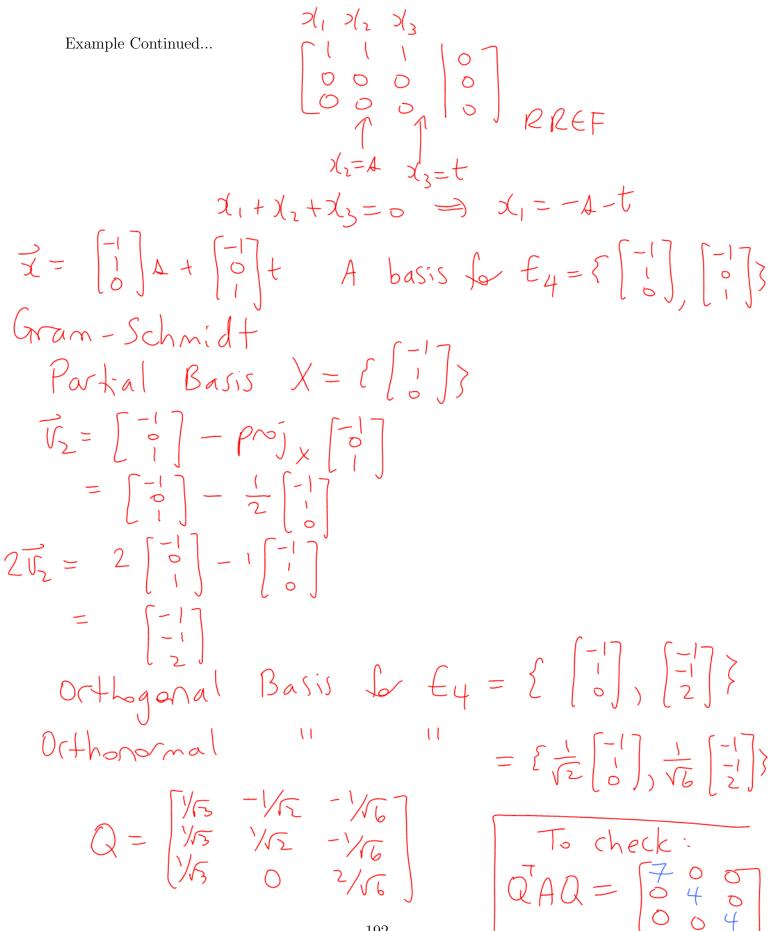
$$A = A$$

**Example:** Is A orthogonally diagonalizable?



b) 
$$A = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$$
  
 $A^{\top} \neq A$   
 $N_{\heartsuit}$ 

**Example:** The matrix  $A = \begin{bmatrix} 5 & 1 & 1 \\ 1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$  has eigenvalues  $\lambda = 4$  and  $\lambda = 7$ . Find Q that orthogonally diagonalizes  $\overline{A}$ To find Q: find an orthonormal basis beach eigenspace. CA-7IGT  $\lambda = 7$  $\begin{bmatrix} -2 & 1 & | & 0 \\ 1 & -2 & 1 & | & 0 \\ 1 & 1 & -2 & 0 \end{bmatrix}$   $\begin{array}{c} x_{1} & x_{2} & x_{3} \\ 1 & 0 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$   $\begin{array}{c} RREF \\ T \\ \end{array}$  $\chi'_{3}=t$  $\chi'_{1}-\chi'_{3}=0 \implies \chi'_{1}=t$  $\chi'_{2}-\chi'_{3}=0 \implies \chi'_{2}=t$  $\overline{\chi} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} t$  A basis for  $\overline{E}_{\tau} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ An orthonormal basis for  $E_7 = C_{\overline{13}} | i ] 3$ [A-4][] 1-4: 



We're going to recap the outer product expansion of AB from Section 3.1.

Example: Find 
$$\begin{bmatrix} 1\\3\\4 \end{bmatrix} \begin{bmatrix} 5 & 6\\7 & 8 \end{bmatrix}$$
 using the outer product expansion.  

$$= \begin{bmatrix} 1\\3 \end{bmatrix} \begin{bmatrix} 5 & 6 \end{bmatrix} + \begin{bmatrix} 2\\4 \end{bmatrix} \begin{bmatrix} 7 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 6\\15 & (8 \end{bmatrix} + \begin{bmatrix} (4 & 16)\\28 & 32 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 22\\43 & 50 \end{bmatrix}$$

Example: Find  $\begin{bmatrix} -1\\ 2\\ 9\\ 3 \end{bmatrix} \begin{bmatrix} 4 & 3\\ 2 & 1 \end{bmatrix}$  using the outer product expansion.  $= \begin{bmatrix} -1\\ 2\\ 2 \end{bmatrix} \begin{bmatrix} 4 & 3\\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 9\\ 3\\ 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix}$   $= \begin{bmatrix} -4\\ -3\\ 8 & 6 \end{bmatrix} + \begin{bmatrix} 18\\ 6 & 3 \end{bmatrix}$   $= \begin{bmatrix} 14\\ 6 & 3 \end{bmatrix}$  **Definition:** Let *A* be a symmetric  $n \times n$  matrix. Let  $\vec{q_1}, \vec{q_2}, \ldots, \vec{q_n}$  be <u>orthonormal</u> eigenvectors written as <u>columns</u>. Let  $\lambda_1, \lambda_2, \ldots, \lambda_n$  be the corresponding eigenvalues. The **spectral decomposition of** *A* is:  $A = \lambda_1 \vec{q_1} \vec{q_1}^T + \lambda_2 \vec{q_2} \vec{q_2}^T + \ldots + \lambda_n \vec{q_n} \vec{q_n}^T$ 

Example: Find a 
$$3 \times 3$$
 matrix  $A$  with eigenvalues  $\lambda = 2$  and  $\lambda = 3$  so that:  
 $E_2 = \operatorname{span}\begin{pmatrix} 1\\1\\1 \end{pmatrix}, \begin{pmatrix} 1\\-1\\0 \end{pmatrix}$  and  $E_3 = \operatorname{span}\begin{pmatrix} 1\\1\\-2 \end{pmatrix}$ .  
O(the genal eigenvectors  $-\frac{1}{\sqrt{3}}\begin{bmatrix} 1\\-1\\\sqrt{3}\end{bmatrix}, -\frac{1}{\sqrt{3}}\begin{bmatrix} -\frac{1}{\sqrt{3}}\\-\frac{1}{\sqrt{3}}\end{bmatrix}, -\frac{1}{\sqrt{3}}\end{bmatrix}, -\frac{1}{\sqrt{3}}\end{bmatrix},$ 

$$\begin{array}{rcl} A = & \lambda_{1} & \overline{q}_{1} & \overline{q}_{1} & + & \lambda_{2} & \overline{q}_{2} & \overline{q}_{2}^{T} & + & \lambda_{3} & \overline{q}_{3} & \overline{q}_{3}^{T} \\ & = & 2 & \frac{1}{2} & \frac{1}{2} & \left[ \frac{1}{1} \right] \begin{bmatrix} 1 & 1 & 1 \right] & + & 2 & \left[ \frac{1}{1} \right] \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \\ & & \overline{x}_{2} & \overline{x}_{2} & \overline{x}_{2} & \left[ \frac{1}{2} \right] \begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \\ & & + & \frac{3}{6} & \left[ \frac{1}{2} & \frac{1}{2} & -2 \\ 0 & 0 & 0 & 0 \\ & & + & \frac{3}{6} & \left[ \frac{1}{2} & \frac{1}{2} & -2 \\ 0 & 0 & 0 & 0 \\ & & + & \frac{3}{6} & \left[ \frac{1}{2} & \frac{1}{2} & -2 \\ 0 & 0 & 0 & 0 \\ & & & + & \frac{3}{6} & \left[ \frac{1}{2} & \frac{1}{2} & -2 \\ 0 & 0 & 0 & 0 \\ & & & + & \frac{3}{6} & \left[ \frac{1}{2} & \frac{1}{2} & -2 \\ 0 & 0 & 0 & 0 \\ & & & & -2 & -2 & 4 \end{array} \right] \end{array}$$

Example Continued...