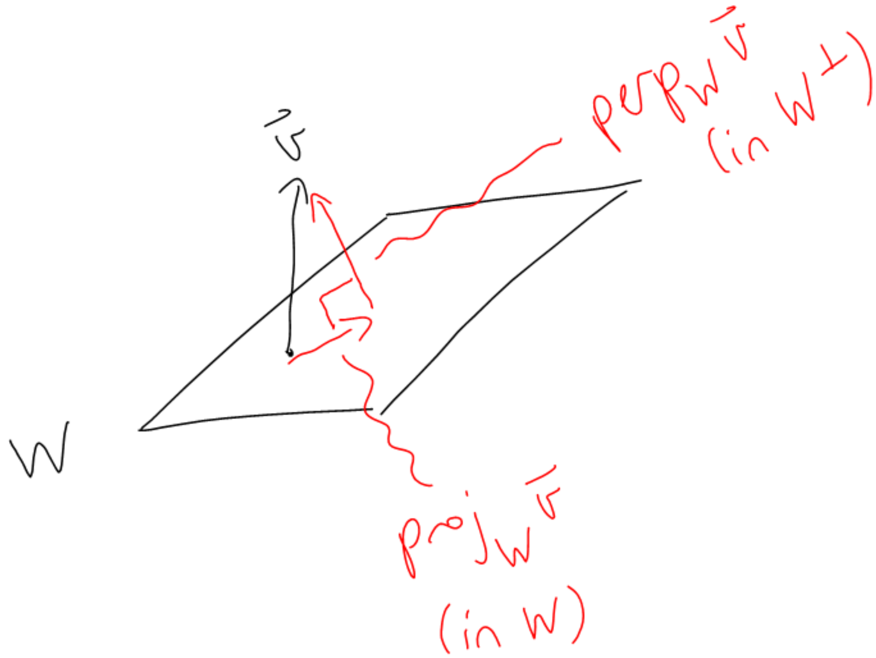
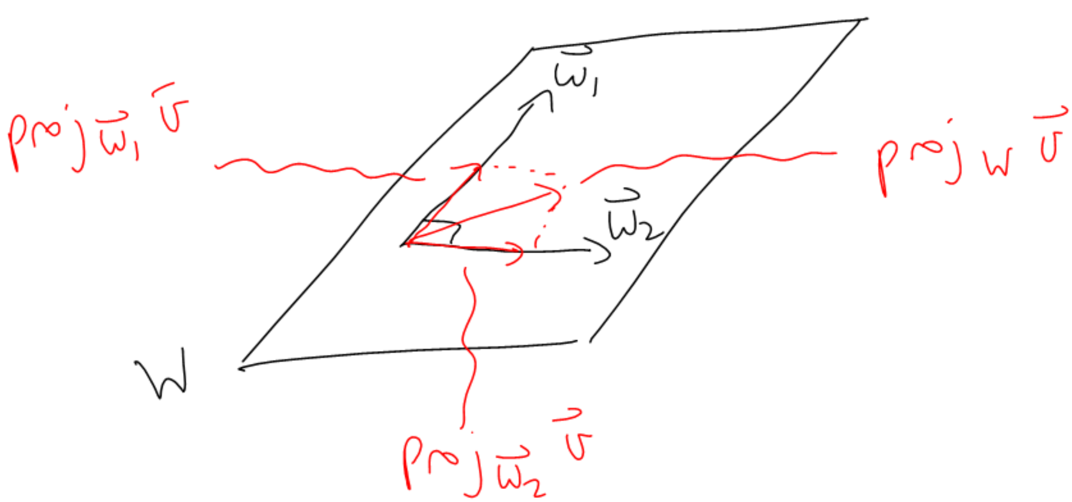


Given an orthogonal basis for W : $\{\vec{w}_1, \vec{w}_2\}$



$$\text{proj}_W \vec{v} = \text{proj}_{\vec{w}_1} \vec{v} + \text{proj}_{\vec{w}_2} \vec{v}$$

5.3 The Gram-Schmidt Procedure

Example: Let $W = \text{span}\left(\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\4\\5\end{bmatrix}, \begin{bmatrix}1\\-3\\-4\\-2\end{bmatrix}\right)$. Find an orthogonal basis for W.

$$\vec{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

Partial Basis $X = \left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}$

$$\vec{v}_2 = \begin{bmatrix}1\\2\\4\\5\end{bmatrix} - \text{proj}_X \begin{bmatrix}1\\2\\4\\5\end{bmatrix}$$

$\boxed{\text{Subtracts off the part of } \begin{bmatrix}1\\2\\4\\5\end{bmatrix} \text{ that is parallel to } \begin{bmatrix}1\\1\\1\\1\end{bmatrix}.}$

$$= \begin{bmatrix}1\\2\\4\\5\end{bmatrix} - \frac{\overset{3}{\cancel{12}}}{\cancel{4}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$= \begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}$$

Partial Basis $X = \left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}\right\}$

$$\vec{v}_3 = \begin{bmatrix}1\\-3\\-4\\-2\end{bmatrix} - \text{proj}_X \begin{bmatrix}1\\-3\\-4\\-2\end{bmatrix}$$

$$= \begin{bmatrix}1\\-3\\-4\\-2\end{bmatrix} - \text{proj}_{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}} \begin{bmatrix}1\\-3\\-4\\-2\end{bmatrix} - \text{proj}_{\begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}} \begin{bmatrix}1\\-3\\-4\\-2\end{bmatrix}$$

$$= \begin{bmatrix}1\\-3\\-4\\-2\end{bmatrix} - \frac{(-8)}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} - \frac{(-7)}{10}\begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}$$

Example Continued...

Can scale $\vec{v}_3$

$$10\vec{v}_3 = 10\begin{bmatrix}1\\-3\\-4\\-2\end{bmatrix} + 20\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + 7\begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}$$

$$= \begin{bmatrix}16\\-17\\-13\\14\end{bmatrix}$$

<u>Orthogonal</u> Basis for $W = \left\{ \begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}, \begin{bmatrix}16\\-17\\-13\\14\end{bmatrix} \right\}$

Comment: This procedure is called **Gram-Schmidt Orthogonalization**.

Example: Modify the basis above to create an orthonormal basis for W.

$$\left\{ \frac{1}{\sqrt{4}}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix}-2\\-1\\1\\2\end{bmatrix}, \frac{1}{\sqrt{910}}\begin{bmatrix}16\\-17\\-13\\14\end{bmatrix} \right\}$$

Example: Find an orthogonal basis for $\mathbb{R}^3$ containing $\begin{bmatrix}1\\1\\5\end{bmatrix}$.

Start with any basis for $\mathbb{R}^3$ containing $\begin{bmatrix}1\\1\\5\end{bmatrix}$,

say $\left\{ \begin{bmatrix}1\\1\\5\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$.

It's a basis for $\mathbb{R}^3$ because

$$\begin{vmatrix}1 & 1 & 0\\1 & 0 & 1\\5 & 0 & 0\end{vmatrix} \neq 0$$

(Fundamental Theorem of Invertible Matrices Section 3.5)

Gram-Schmidt:

Partial Basis $X = \left\{ \begin{bmatrix}1\\1\\5\end{bmatrix} \right\}$

$$\vec{v}_2 = \begin{bmatrix}1\\0\\0\end{bmatrix} - \operatorname{proj}_X \begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$= \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{27}\begin{bmatrix}1\\1\\5\end{bmatrix}$$

$$27\vec{v}_2 = 27\begin{bmatrix}1\\0\\0\end{bmatrix} - 1\begin{bmatrix}1\\1\\5\end{bmatrix}$$

$$= \begin{bmatrix}26\\-1\\-5\end{bmatrix}$$

Partial Basis $X = \left\{ \begin{bmatrix}1\\1\\5\end{bmatrix}, \begin{bmatrix}26\\-1\\-5\end{bmatrix} \right\}$

Example Continued...

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \text{proj}_X \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \text{proj}_{\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \text{proj}_{\begin{bmatrix} 26 \\ -1 \\ -5 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{27} \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} - \frac{(-1)}{702} \begin{bmatrix} 26 \\ -1 \\ -5 \end{bmatrix}$$

$$702\,\vec{v}_3 = 702 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 26 \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} + 1 \begin{bmatrix} 26 \\ -1 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 675 \\ -135 \end{bmatrix}$$

Orthogonal Basis for $\mathbb{R}^3$

$$= \left\{ \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 26 \\ -1 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 675 \\ -135 \end{bmatrix} \right\}$$

Definition: Let A be a matrix with linearly independent columns.
The **QR Factorization of** A is:
$A = QR$ where Q is an orthogonal matrix and R is upper triangular.

orthonormal columns

Example: Let $A = QR$ for an orthogonal matrix Q. Show that $R = Q^T A$.

$$A = QR$$

Left-multiply by Q^T:

$$Q^T A = \underbrace{Q^T Q}_{I} R$$

$$Q^T A = IR$$

$$Q^T A = R$$

Fact: Let $A = QR$ for an orthogonal matrix Q.
To find Q: Apply Gram-Schmidt to the columns of A, and normalize.
Then $R = Q^T A$.

Example: Find Q and R for $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$.

Q: Gram-Schmidt on columns of A
and normalize

Partial Basis $X = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \text{proj}_X \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} - \cancel{\frac{1}{1}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

Partial Basis $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\}$

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \text{proj}_X \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \text{proj}_{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \text{proj}_{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} - \frac{0}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \frac{3}{5} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

Example Continued...

$$5\vec{v}_3 = 5\begin{bmatrix}0\\1\\1\\3\end{bmatrix} - 3\begin{bmatrix}0\\1\\2\\0\end{bmatrix}$$

$$= \begin{bmatrix}0\\2\\-1\\15\end{bmatrix}$$

$$\text{Orthogonal Basis} = \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\2\\-1\\15\end{bmatrix} \right\}$$

$$\text{Orthonormal Basis} = \left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix}0\\1\\2\\0\end{bmatrix}, \frac{1}{\sqrt{230}}\begin{bmatrix}0\\2\\-1\\15\end{bmatrix} \right\}$$

$$Q = \begin{bmatrix}1 & 0 & 0\\ 0 & 1/\sqrt{5} & 2/\sqrt{230}\\ 0 & 2/\sqrt{5} & -1/\sqrt{230}\\ 0 & 0 & 15/\sqrt{230}\end{bmatrix}$$

$$R = Q^T A$$

$$= \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1/\sqrt{5} & 2/\sqrt{5} & 0\\ 0 & 2/\sqrt{230} & -1/\sqrt{230} & 15/\sqrt{230}\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 2 & 1\\ 0 & 0 & 3\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1 & 0\\ 0 & 5/\sqrt{5} & 3/\sqrt{5}\\ 0 & 0 & 46/\sqrt{230}\end{bmatrix}$$