Example: Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \\ 0 & 0 \end{bmatrix}$ and let $W = \text{col}(A)$. Find a basis for $W^\perp$.

$$W = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \right)$$

Solve $\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 3 & 4 & 0 & 0 \end{array}\right]$

$$\rightsquigarrow \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] \end{array} \text{ RREF}$$

$z = t$

$x = 0$

$y = 0$

$\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t$ Basis for $W^\perp = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

More formally: $W = \text{Col}(A)$

$W = \text{row}(A^T)$

$W^\perp = \text{null}(A^T)$

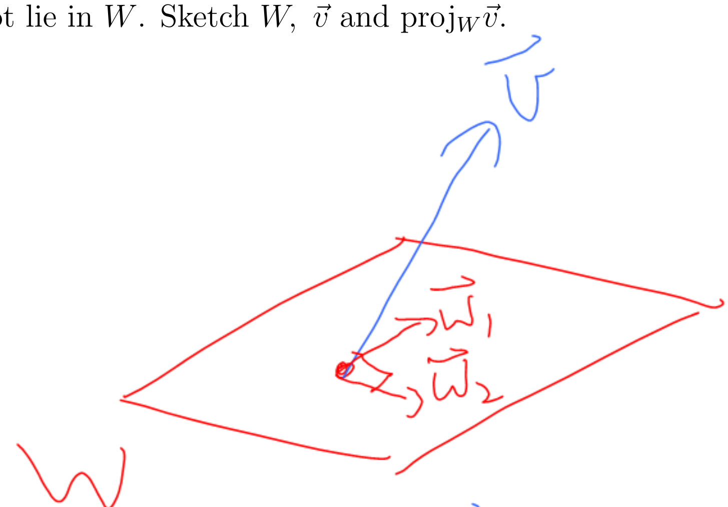
Fact: For any matrix A, $[\text{col}(A)]^\perp = \text{null}(A^T)$.

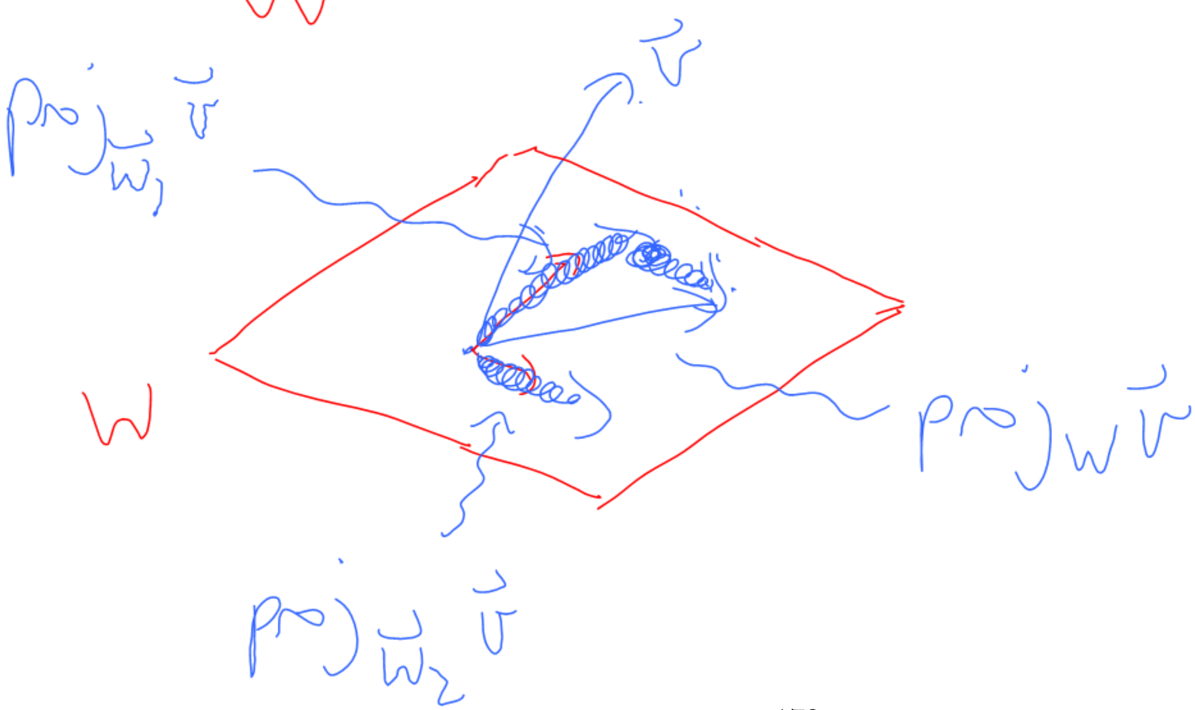
Definition: Let W be a subspace of $\mathbb{R}^n$ with orthogonal basis $\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_k\}$.
The **orthogonal projection of** $\vec{v}$ onto W is:

$\text{proj}_W \vec{v} = \text{proj}_{\vec{w}_1}\vec{v} + \text{proj}_{\vec{w}_2}\vec{v} + \ldots + \text{proj}_{\vec{w}_k}\vec{v}$.

Comment: This formula only applies when the basis for W is **orthogonal**.

Example: Let W be a plane through the origin in $\mathbb{R}^3$. Let $\vec{v}$ be a vector in $\mathbb{R}^3$ that does not lie in W. Sketch W, $\vec{v}$ and $\text{proj}_W \vec{v}$.





$\vec{w}_1$

$\vec{w}_2$

Example: W has orthogonal basis $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \}$.

Find the orthogonal projection of $\vec{v} = \begin{bmatrix} 1 \\ 5 \\ -3 \\ 7 \end{bmatrix}$ onto W.

Orthogonal basis $\Rightarrow$

$$\text{proj}_W \vec{v} = \text{proj}_{\vec{w}_1} \vec{v} + \text{proj}_{\vec{w}_2} \vec{v}$$

$$= \frac{\vec{w}_1 \cdot \vec{v}}{\|\vec{w}_1\|^2} \vec{w}_1 + \ldots$$

$$= \frac{1}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \frac{8}{2} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 4 \\ -4 \\ 0 \end{bmatrix}$$

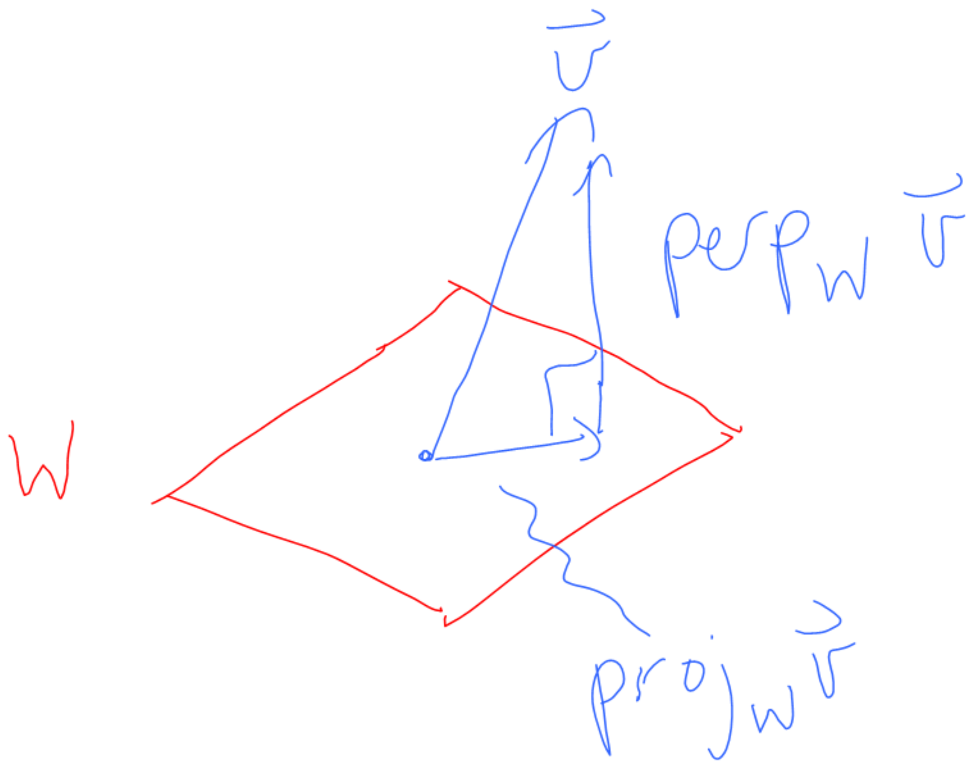
Definition: The **orthogonal decomposition** of $\vec{v}$ with respect to W is:

$\vec{v} = \text{proj}_W \vec{v} + \boxed{\text{perp}_W \vec{v}}$

where $\text{proj}_W \vec{v}$ is in W and $\text{perp}_W \vec{v}$ is in $W^{\perp}$.

"perp with respect to W of $\vec{v}$"

Example: Let W be a plane through the origin in $\mathbb{R}^3$. Let $\vec{v}$ be a vector in $\mathbb{R}^3$ that does not lie in W. Sketch W, $\vec{v}$, $\text{proj}_W \vec{v}$ and $\text{perp}_W \vec{v}$.



$\vec{w}_1$ $\vec{w}_2$

Example: W has orthogonal basis $\mathcal{B} = \{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \}$.

Find the orthogonal decomposition of $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ with respect to W.

Find $\text{proj}_W \vec{v}$ and $\text{perp}_W \vec{v}$.

Orthogonal basis

$$\Rightarrow \text{proj}_W \vec{v} = \text{proj}_{\vec{w}_1} \vec{v} + \text{proj}_{\vec{w}_2} \vec{v}$$

$$= \frac{\vec{w}_1 \cdot \vec{v}}{\|\vec{w}_1\|^2} \vec{w}_1 + \ldots$$

$$= \frac{-4}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{11}{9} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$= \frac{-18}{9} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \frac{11}{9} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

$$= \frac{1}{9} \begin{bmatrix} 4 \\ -11 \\ 40 \end{bmatrix}$$

$$\vec{v} = \text{proj}_W \vec{v} + \text{perp}_W \vec{v}$$

$$\text{perp}_W \vec{v} = \vec{v} - \text{proj}_W \vec{v} = \frac{9}{9} \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix} - \frac{1}{9} \begin{bmatrix} 4 \\ -11 \\ 40 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 5 \\ 20 \\ 5 \end{bmatrix}$$