

Test Review

$$\textcircled{1} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 0 & 0 \end{bmatrix} \text{ has RREF} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{bmatrix} \text{ has RREF} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Find a basis for:

a) $\text{col}(A)$

b) $\text{row}(A)$, consisting of rows of A

c) $\text{null}(A)$

a) Use columns 1 and 2 of A

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\}$$

b) $\text{row}(A) = \text{col}(A^T)$

Use columns 1 and 3 of A^T

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

or $\left\{ [1 \ 1 \ 1], [1 \ 0 \ 0] \right\}$

c) Solve $A\vec{x} = \vec{0}$

$$[A \mid \vec{0}]$$

$$[\text{RREF of } A \mid \vec{0}]$$

$$\begin{array}{ccc} x & y & z \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \end{array}$$



$$z = t$$

$$x = 0$$

$$y + z = 0 \Rightarrow y = -t$$

$$\text{null}(A) \longrightarrow \vec{x} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} t$$

$$\text{Basis for null}(A) = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

② Compute $\det A$ for $A = \begin{bmatrix} 1 & 2 & -3 \\ 9 & 8 & 7 \\ 4 & -5 & 4 \end{bmatrix}$

Cofactor expansion

$$|A| = 1 \begin{vmatrix} 8 & 7 \\ -5 & 4 \end{vmatrix} - 2 \begin{vmatrix} 9 & 7 \\ 4 & 4 \end{vmatrix} - 3 \begin{vmatrix} 9 & 8 \\ 4 & -5 \end{vmatrix}$$

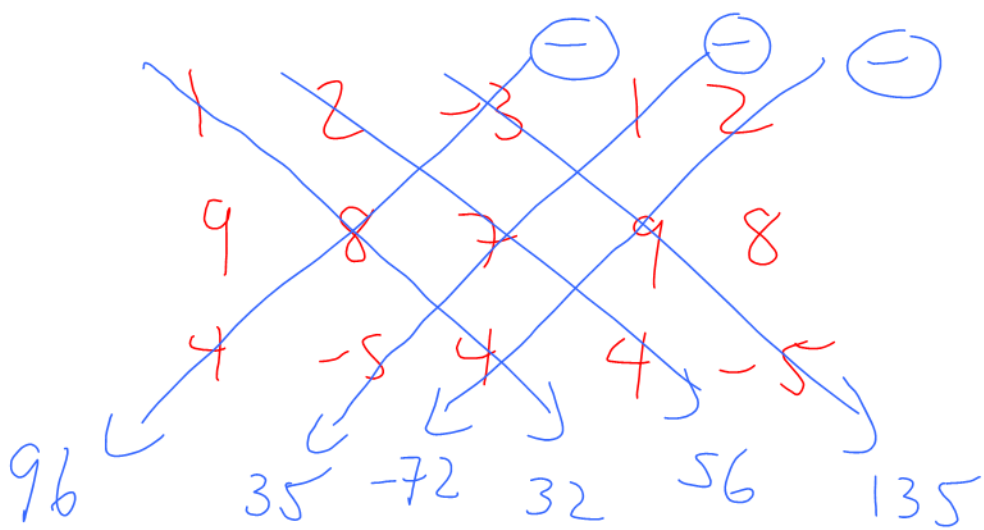
$$= 1(67) - 2(8) - 3(-77)$$

$$= 282$$

$$\begin{bmatrix} + & - & + \\ - & + & - \end{bmatrix}$$

OR

Quick Method



$$|A| = 96 + 35 - 72 + 32 + 56 + 135$$

$$= 282$$

③ Solve using Cramer's Rule

$$\begin{cases} 4x - 3y = 12 \\ 8x + 6y = 16 \end{cases}$$

$$|A| = \begin{vmatrix} 4 & -3 \\ 8 & 6 \end{vmatrix} = 48$$

$$|A_1| = \begin{vmatrix} 12 & -3 \\ 16 & 6 \end{vmatrix} = 120$$

$$|A_2| = \begin{vmatrix} 4 & 12 \\ 8 & 16 \end{vmatrix} = -32$$

$$x = \frac{|A_1|}{|A|} = \frac{120}{48} = \frac{5}{2} \quad y = \frac{|A_2|}{|A|} = \frac{-32}{48} = -\frac{2}{3}$$

④ a) Find all the eigenvalues of $A = \begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -4 \\ 3 & -1-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)(-1-\lambda) + 12 = 0$$

$$-6 - 6\lambda + \lambda + \lambda^2 + 12 = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda = 2, 3$$

b) Find all eigenvectors of

$A = \begin{bmatrix} 6 & -4 \\ 3 & -1 \end{bmatrix}$ corresponding to $\lambda = 3$.

Solve $[A - \lambda I | \vec{0}]$

$$[A - 3I | \vec{0}]$$

$$\begin{bmatrix} 3 & -4 & | & 0 \\ 3 & -4 & | & 0 \end{bmatrix}$$

$$\frac{R_1}{3}$$

$$\begin{bmatrix} 1 & -\frac{4}{3} & | & 0 \\ 3 & -4 & | & 0 \end{bmatrix}$$

$$R_2 - 3R_1 \quad \begin{bmatrix} x_1 & x_2 \\ 1 & -\frac{4}{3} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \quad \text{RREF}$$

\uparrow
 $x_2 = t$

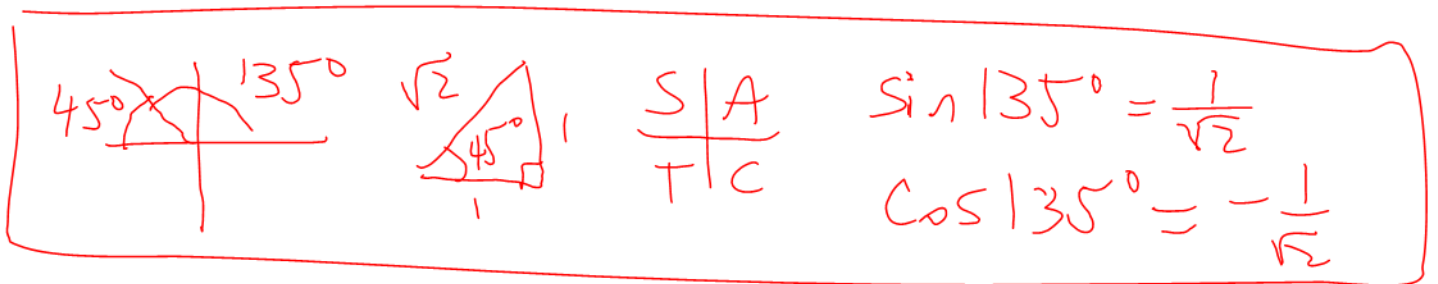
$$x_1 - \frac{4}{3}x_2 = 0 \Rightarrow x_1 = \frac{4}{3}t$$

$$\vec{x} = \begin{bmatrix} 4/3 \\ 1 \end{bmatrix} t \quad (t \neq 0)$$

$$\text{OR } \vec{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} t \quad (t \neq 0)$$

⑤ Find the standard matrix for $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ where:
 T first rotates a vector by 135° then projects it on the line through the origin with $\vec{d} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

$$[T_1] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \theta = 135^\circ = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$



$$[T_2] = \frac{1}{a^2 + b^2} \begin{bmatrix} a^2 & ab \\ ab & b^2 \end{bmatrix} \quad \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$$

$$[T] = [T_2][T_1]$$

$$= \frac{1}{5\sqrt{2}} \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{5\sqrt{2}} \begin{bmatrix} -6 & -2 \\ 3 & 1 \end{bmatrix}$$

In general

$$[T] = \dots [T_2][T_1]$$

⑥ a) Find the LU Factorization of

$$A = \begin{bmatrix} 3 & 2 & -4 \\ -27 & -12 & 43 \\ 21 & 26 & -22 \end{bmatrix}$$

$$\begin{array}{l} R_2 + 9R_1 \\ R_3 - 7R_1 \end{array} \begin{bmatrix} 3 & 2 & -4 \\ 0 & 6 & 7 \\ 0 & 12 & 6 \end{bmatrix} \begin{array}{l} \\ k = -9 \\ k = 7 \end{array}$$

$$R_3 - 2R_2 \begin{bmatrix} 3 & 2 & -4 \\ 0 & 6 & 7 \\ 0 & 0 & -8 \end{bmatrix} \begin{array}{l} \\ \\ k = 2 \end{array}$$

u

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -9 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix}$$

b) Solve $A\vec{u} = \begin{bmatrix} -5 \\ 78 \\ 7 \end{bmatrix}$
using part a).

$$L \cancel{U} x = \vec{b}$$

\downarrow
 \vec{y}

$$1) L \vec{y} = \vec{b}$$

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ 1 & 0 & 0 \\ -9 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} \left| \begin{array}{l} -5 \\ 78 \\ 7 \end{array} \right. \begin{array}{l} y_1 = -5 \\ y_2 = 33 \\ y_3 = -24 \end{array}$$

$$2) U \vec{x} = \vec{y}$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ 3 & 2 & -4 \\ 0 & 6 & 7 \\ 0 & 0 & -8 \end{bmatrix} \left| \begin{array}{l} -5 \\ 33 \\ -24 \end{array} \right. \begin{array}{l} x_1 = 1 \\ x_2 = 2 \\ x_3 = 3 \end{array}$$