Test 3

FRI MAR 22

3.4–3.6 , 4.1–4.2

Definition: An **orthogonal matrix** Q is an $n \times n$ matrix whose columns form an orthonormal set. For example, the following matrix is orthogonal:

!

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left\| \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix} \right\| = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = 1$$

Fact: A square matrix Q is orthogonal if and only if $Q^T Q = I$.

Example: Verify that $Q^T Q = I$ for $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$Q^T Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

ones indicate columns of Q have length 1

zeros indicate columns of Q are orthogonal

$$= I$$

Fact: If Q is orthogonal then $Q^{-1} = Q^T$.

Example: Prove the fact above.

Q orthogonal

$\Longrightarrow \quad Q^T Q = I$

Right-multiply by Q^{-1}: $\quad Q^T \underbrace{\cancel{Q Q^{-1}}}_{I} = I Q^{-1}$

$Q^T = Q^{-1}$

Example: Let Q be an orthogonal matrix. Show that Q^{-1} is orthogonal.

To show M is orthogonal : $\quad M^T M = I$

$\quad\quad\;\; \| \quad\quad Q^{-1} \quad\quad \| \quad\quad\quad : \quad (Q^{-1})^T Q^{-1} = I$

$$
\begin{aligned}
(Q^{-1})^T Q^{-1} &= (Q^T)^T Q^{-1} \quad (Q \text{ is orthogonal}) \\
&= Q Q^{-1} \\
&= I \;\checkmark
\end{aligned}
$$

Example: Determine all values of x, y and z so that $\begin{bmatrix} \frac{1}{2} & y \\ x & z \end{bmatrix}$ is an orthogonal matrix.

$$\left\| \begin{bmatrix} \frac{1}{2} \\ x \end{bmatrix} \right\| = 1$$

$$\sqrt{\frac{1}{4} + x^2} = 1$$

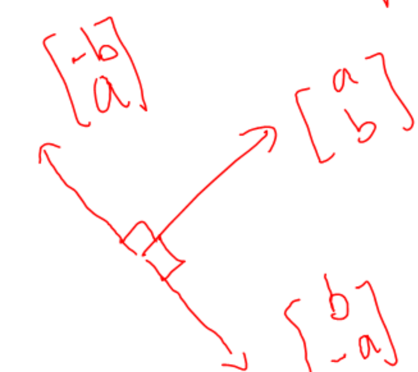
$$\frac{1}{4} + x^2 = 1$$

$$x^2 = \frac{3}{4}$$

$$x = \pm\sqrt{\frac{3}{4}}$$

$$x = \pm\frac{\sqrt{3}}{2}$$

If the 1st column is $\begin{bmatrix} a \\ b \end{bmatrix}$ then there are 2 options for 2nd column:



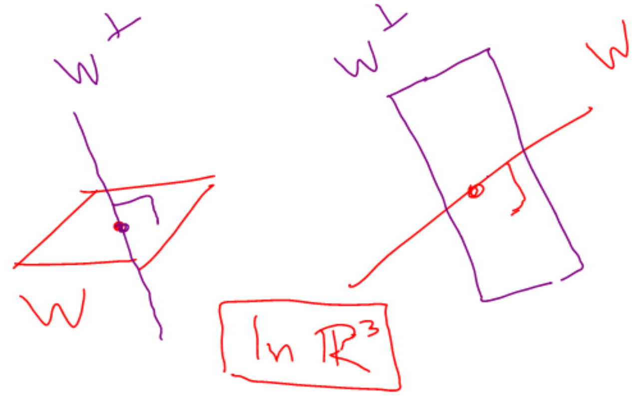
(all vectors have length 1)

$x = \frac{\sqrt{3}}{2}$: $\frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$ $\quad \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}$

$x = -\frac{\sqrt{3}}{2}$: $\frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$ $\quad \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{bmatrix}$

5.2 Orthogonal Complements and Projections

Throughout Chapter 5, W will represent a subspace of $\mathbb{R}^n$. Rephrased: W is the span of one or more vectors in $\mathbb{R}^n$.



Definition: The **orthogonal complement of** W is:
$W^\perp = \{\vec{v} \text{ in } \mathbb{R}^n \text{ such that } \vec{v}\cdot\vec{w} = 0 \text{ for all } \vec{w} \text{ in } W\}$.
$W^\perp$ is pronounced "W perp".

Example: Let $W = \text{span}(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix})$. Find $W^\perp$.

$$W^\perp = \left\{ \begin{bmatrix}x\\y\\z\end{bmatrix} \text{ such that } \begin{bmatrix}x\\y\\z\end{bmatrix}\cdot\begin{bmatrix}1\\0\\0\end{bmatrix} = 0 \text{ and } \begin{bmatrix}x\\y\\z\end{bmatrix}\cdot\begin{bmatrix}1\\1\\3\end{bmatrix} = 0 \right\}$$

$$= \left\{ \begin{bmatrix}x\\y\\z\end{bmatrix} \text{ such that } x = 0 \text{ and } x + y + 3z = 0 \right\}$$

Shortcut

Solve
$$\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right] \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right] \quad \text{RREF}$$

$z = t$

$x = 0$

$y + 3z = 0 \Rightarrow y = -3t$

$\vec{x} = \begin{bmatrix}0\\-3\\1\end{bmatrix} t \qquad W^\perp = \text{span}(\begin{bmatrix}0\\-3\\1\end{bmatrix}) \qquad$ A basis for $W^\perp = \{\begin{bmatrix}0\\-3\\1\end{bmatrix}\}$

Recall that the **dimension** of a subspace W is the number of vectors in a basis for W.

(Section 3.5)

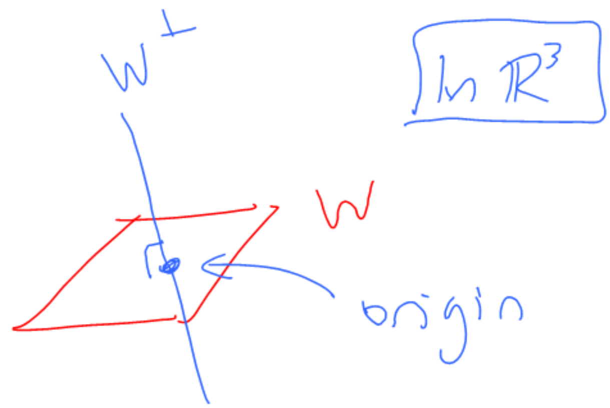
Three Facts about $W^{\perp}$

For any subspace W of $\mathbb{R}^n$:

1) $\dim W + \dim W^{\perp} = n$

2) $W \cap W^{\perp} = \{\vec{0}\}$

3) $(W^{\perp})^{\perp} = W$



Example: Let $W = \text{span}(\underbrace{\begin{bmatrix} 1 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}}_{\text{not multiples}})$. Find the dimension of W and $W^{\perp}$.

$\dim W = 2$

$\dim W + \dim W^{\perp} = 5$

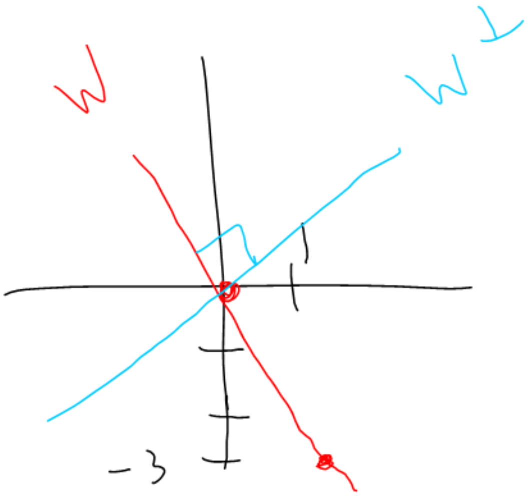
$\dim W^{\perp} = 3$

Example: Let $W = \{\begin{bmatrix} x \\ y \end{bmatrix}$ such that $3x + y = 0\}$. Find a basis for W and for $W^{\perp}$.

2 points on W: $P = (0,0)$ $Q = (1,-3)$

Direction vector for W: $\vec{d} = \overrightarrow{PQ} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

A basis for $W = \left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix} \right\}$



$W^{\perp}$:

$$\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \left[\begin{array}{cc|c} 1 & -3 & 0 \end{array}\right] \end{array}$$

$y = t$

$x - 3y = 0 \Rightarrow x = 3t$

$\vec{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} t$

A basis for $W^{\perp} = \left\{ \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

Example: Let $A = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 6 \end{bmatrix}$ and let $W = \text{row}(A)$. Find a basis for $W^{\perp}$.

$$W = \text{span}\left([1\ 0\ 0\ 4],\ [0\ 1\ 1\ 6] \right)$$

Solve
$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & & \\ \hline 1 & 0 & 0 & 4 & | & 0 \\ 0 & 1 & 1 & 6 & | & 0 \end{array}$$
RREF

$x_3 = s \qquad x_4 = t$

$$x_1 + 4x_4 = 0 \Rightarrow x_1 = -4t$$
$$x_2 + x_3 + 6x_4 = 0 \Rightarrow x_2 = -s - 6t$$

$$\vec{x} = \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -4 \\ -6 \\ 0 \\ 1 \end{bmatrix} t$$

Basis for $W^{\perp} = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -6 \\ 0 \\ 1 \end{bmatrix} \right\}$

More Formally: $W^{\perp} = \{ \vec{x}$ such that $\vec{x}$ is orthogonal to each row of $A \}$

$= \left\{ \vec{x} \text{ such that } \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 1 & 6 \end{bmatrix} \vec{x} = \vec{0} \right\}$

$= \text{null}(A)$

Fact: For any matrix A, $[\text{row}(A)]^{\perp} = \text{null}(A)$.