Test

FRI MAR 22

3.4–3.6, 4.1–4.2

(6 Questions)

Bring: calculator

music/earplugs

Practice Questions on Webpage

Chapter 5: Orthogonality

5.1 Orthogonality

Definition: An **orthogonal set** is a set of two or more vectors such that any two of the vectors are orthogonal.

Example: Verify that $\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \}$ is an orthogonal set.

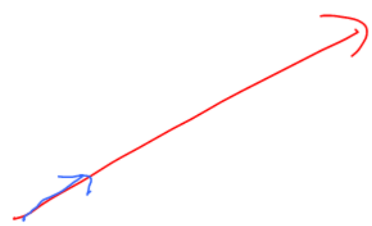
$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = 0$ ✓ $\quad \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 0$ ✓ $\quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 0$ ✓

Definition: To **normalize** a vector means to find a unit vector in the same direction.

Example: Normalize $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

$\|\vec{u}\| = \sqrt{6}$

$\dfrac{\vec{u}}{\|\vec{u}\|} = \dfrac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$



Definition: An **orthonormal set** is an orthogonal set in which all vectors have length 1. For example, the following is an orthonormal set:

$\{ \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \}$.

Fact: A set of n nonzero orthogonal vectors in $\mathbb{R}^n$ forms a basis for $\mathbb{R}^n$.

Comment: This implies that a set of n nonzero orthonormal vectors in $\mathbb{R}^n$ forms a basis for $\mathbb{R}^n$.

Example: Find an orthonormal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ for $\mathbb{R}^3$ such that:
$\vec{u}_1$ is parallel to $[2, 0, 1]$ and $\vec{u}_2$ is parallel to $[1, 3, -2]$.

Direction of $\vec{u}_3 = [2,0,1] \times [1,3,-2]$
$= [-3, 5, 6]$

$$\begin{matrix} 2 & 0 & 1 & 2 & 0 \\ 1 & 3 & -2 & 1 & 3 \end{matrix}$$

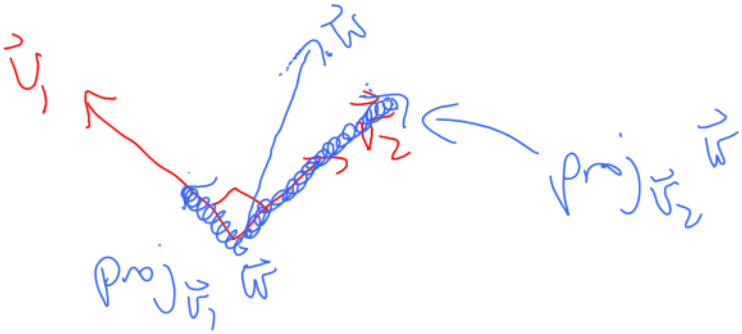
$$\left\{ \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \frac{1}{\sqrt{70}}\begin{bmatrix} -3 \\ 5 \\ 6 \end{bmatrix} \right\}$$
is an orthonormal set.

3 nonzero orthonormal vectors
$\Rightarrow$ the set is a basis for $\mathbb{R}^3$.

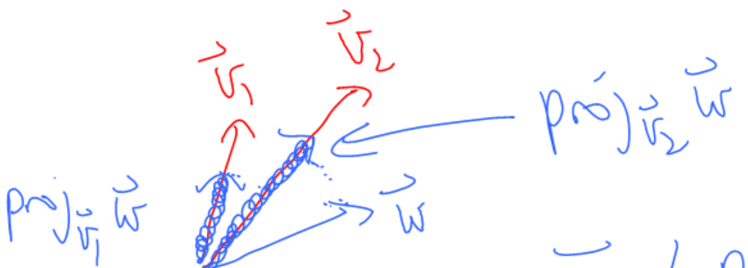
Fact: Suppose $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is an orthogonal basis for $\mathbb{R}^n$.
For any vector $\vec{w}$ in $\mathbb{R}^n$:

$\vec{w} = \text{proj}_{\vec{v}_1}\vec{w} + \text{proj}_{\vec{v}_2}\vec{w} + \ldots + \text{proj}_{\vec{v}_n}\vec{w}$

$\vec{w} = \text{proj}_{\vec{v}_1}\vec{w} + \text{proj}_{\vec{v}_2}\vec{w}$



Example: Draw a sketch to show that $\vec{w} \neq \text{proj}_{\vec{v}_1}\vec{w} + \text{proj}_{\vec{v}_2}\vec{w} + \ldots + \text{proj}_{\vec{v}_n}\vec{w}$
if $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is not orthogonal.



$\vec{w} \neq \text{proj}_{\vec{v}_1}\vec{w} + \text{proj}_{\vec{v}_2}\vec{w}$

Example: $\mathcal{B} = \{ \overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}, \overset{\vec{v}_3}{\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}} \}$ is an orthogonal basis for $\mathbb{R}^3$. Write $\vec{w} = \begin{bmatrix} 5 \\ 0 \\ 9 \end{bmatrix}$ as a linear combination of the basis vectors.

Orthogonal basis

$$\Rightarrow \quad \vec{w} = \operatorname{proj}_{\vec{v}_1} \vec{w} + \operatorname{proj}_{\vec{v}_2} \vec{w} + \operatorname{proj}_{\vec{v}_3} \vec{w}$$

$$= \frac{\vec{v}_1 \cdot \vec{w}}{\|\vec{v}_1\|^2} \vec{v}_1 + \dots$$

$$= \frac{41}{18} \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} + \frac{5}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \frac{1}{9} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \checkmark$$

$$\text{or } \frac{41}{18} \vec{v}_1 + \frac{5}{2} \vec{v}_2 + \frac{1}{9} \vec{v}_3 \checkmark$$

If the basis weren't orthogonal:

Let $c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{w}$

$$\begin{array}{cccc} c_1 & c_2 & c_3 & \\ \end{array}$$
$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & \vec{w} \end{array} \right]$$

Definition: An **orthogonal matrix** Q is an $n \times n$ matrix whose columns form an orthonormal set. For example, the following matrix is orthogonal:

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left\| \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{bmatrix} \right\| = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = 1$$

!

Fact: A square matrix Q is orthogonal if and only if $Q^T Q = I$.

Example: Verify that $Q^T Q = I$ for $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$.