Definition: The **characteristic equation** of A is $|A - \lambda I| = 0$.
The **algebraic multiplicity** of an eigenvalue λ_i is the exponent on $(\lambda_i - \lambda)$ or $(\lambda - \lambda_i)$
in the characteristic equation.
The **geometric multiplicity** of an eigenvalue is the number of basis vectors in the corresponding eigenspace.

Example: Let A have characteristic equation $(7-\lambda)^3(9-\lambda)^2 = 0$. A basis for E_7 consists of one vector. A basis for E_9 consists of two vectors. Find the eigenvalues of A and state their algebraic multiplicities and their geometric multiplicities.

Eigenvalue	Alg. Mult.	Geo. Mult.
$\lambda = 7$	3	1
$\lambda = 9$	2	2

Fact: For each eigenvalue:
$1 \leq$ geometric multiplicity $\leq$ algebraic multiplicity

Comment: If a matrix has
(geometric multiplicity) = (algebraic multiplicity) for all its eigenvalues
then the matrix has a nice property.
We'll see the details in Section 4.4.

We'll look at five properties of eigenvalues.

Property 1: A is invertible if and only if 0 is not an eigenvalue of A.

Example: Prove Property 1.

A is invertible

$\iff \det A \neq 0$

$\iff \det(A - 0I) \neq 0$

$\iff$ 0 is <u>not</u> an eigenvalue of A

Property 2: If A is invertible and $A\vec{x} = \lambda\vec{x}$ then $\vec{x}$ is an eigenvector of A^{-1} with eigenvalue $\frac{1}{\lambda}$.

Example: Prove Property 2.

$A\vec{x} = \lambda\vec{x}$ and A^{-1} exists

$\Rightarrow \quad A^{-1}A\vec{x} = A^{-1}\lambda\vec{x}$

$\Rightarrow \quad \cancel{I}\vec{x} = \lambda A^{-1}\vec{x}$

$\Rightarrow \quad \frac{1}{\lambda}\vec{x} = A^{-1}\vec{x}$ ($\lambda \neq 0$ from Property 1)

$\Rightarrow \quad A^{-1}\vec{x} = \frac{1}{\lambda}\vec{x}$

eigenvector of A^{-1}

eigenvalue of A^{-1}

Property 3: Let n be a non-negative integer. If $A\vec{x} = \lambda\vec{x}$ then $A^n\vec{x} = \lambda^n\vec{x}$.

Example: Prove Property 3.

$$
\begin{aligned}
A^n\vec{x} &= A^{n-1}(A\vec{x}) \\
&= A^{n-1}(\lambda\vec{x}) \\
&= \lambda A^{n-1}\vec{x} \\
&\ \vdots \\
&= \lambda^n\vec{x}
\end{aligned}
$$

eigenvalue of A^n

eigenvector of A^n

Property 4: If $A\vec{x} = \lambda\vec{x}$ then $\vec{x}$ is an eigenvector of $A + kI$ with eigenvalue $\lambda + k$.

Example: Prove Property 4.

$$(A+kI)\vec{x} = A\vec{x} + kI\vec{x}$$
$$= \lambda\vec{x} + k\vec{x}$$
$$= (\lambda + k)\vec{x}$$

eigenvalue of $A+kI$

eigenvector of $A+kI$

Property 5: Let n be a non-negative integer.
Suppose A has eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m$ corresponding to eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. Then:
$A^n(c_1\vec{x}_1 + c_2\vec{x}_2 + \ldots + c_m\vec{x}_m) = c_1\lambda_1^n\vec{x}_1 + c_2\lambda_2^n\vec{x}_2 + \ldots + c_m\lambda_m^n\vec{x}_m$.

Comment: This is a generalization of Property 3.
Note that the coefficients are preserved.

Example: Suppose A has:
eigenvalue $\lambda_1 = -2$ corresponding to eigenvector $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and
eigenvalue $\lambda_2 = 3$ corresponding to eigenvector $\vec{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Calculate $A^3 \begin{bmatrix} 11 \\ 2 \end{bmatrix}$.

1) Let $c_1 \vec{v}_1 + c_2 \vec{v}_2 = \begin{bmatrix} 11 \\ 2 \end{bmatrix}$

$$\begin{array}{cc} c_1 & c_2 \\ \end{array}$$
$$\left[\begin{array}{cc|c} 1 & 2 & 11 \\ 2 & -1 & 2 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 4 \end{array}\right]$$

$c_1 = 3, \ c_2 = 4$

2) $A^3 \begin{bmatrix} 11 \\ 2 \end{bmatrix}$

$= A^3 (3\vec{v}_1 + 4\vec{v}_2)$

$= c_1 \lambda_1^3 \vec{v}_1 + c_2 \lambda_2^3 \vec{v}_2$

$= 3(-8)\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 4(27)\begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= -24 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 108 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$= \begin{bmatrix} 192 \\ -156 \end{bmatrix}$

Example: Suppose A has the eigenvalue 3 corresponding to the eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

List one eigenvector and one eigenvalue for each of the following matrices: $A^{-1}, A^4, A + 2I$.

(Properties 2-4)

Matrix	Eigenvector	Eigenvalue
A^{-1}	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$\frac{1}{3}$
A^4	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$3^4 = 81$
$A+2I$	$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$	$3+2=5$

4.4 Diagonalization

Definition: An $n \times n$ matrix A is **diagonalizable** if there exist an invertible matrix P and a diagonal matrix D so that $P^{-1}AP = D$.

Fact: To find P we find a basis for each eigenspace of A. The basis vectors go into the columns of P. The matrix D has the eigenvalues on the diagonal, in the same order as P.

Example: Let $A = \begin{bmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Find P and D that diagonalize A.

Eigenvalues of A: $\lambda = 2, 3$

(A is upper triangular)

Eigenspace E_2:

$$\left[A - \lambda I \mid \vec{0}\right]$$

$$\left[A - 2I \mid \vec{0}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Reorder rows $\left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right]$

$R_3 + 2R_2$ $\begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$ RREF

$x_1 = t$, $x_2 = 0$, $x_3 = 0$

$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t$ Basis for $E_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

Example Continued...

Eigenspace E_3:

$$[A-3I\,|\,\vec{0}]$$

$$\left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \text{ RREF}$$

$x_2 = s$ $x_3 = t$

$$x_1 + 2x_3 = 0 \Rightarrow x_1 = -2t$$

$$\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} t$$

Basis for $E_3 = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$P = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

basis vectors in columns

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

eigenvalues, in same order as P

Check: $P^{-1}AP = D$ ✓ (P^{-1} is guaranteed to exist if P exists)