Test 3

FRI MAR 22

3.4–3.6, 4.1–4.2

(6 Questions)

Bring calculator

Bring music/earplugs

Practice Problems on website

Fact: Cramer's Rule
Let A be an $n \times n$ matrix. When $\det A \neq 0$, the system $A\vec{x} = \vec{b}$ has a unique solution:
i-th variable= $\frac{|A_i|}{|A|}$
where $A_i = A$ with the i-th column replaced by $\vec{b}$.

Example: Solve using Cramer's Rule:

$$A = \begin{bmatrix} 2 & 3 & 2 \\ 3 & 0 & 5 \\ 4 & 1 & 1 \end{bmatrix}$$

$$\begin{aligned} 2x + 3y + 2z &= -11 \\ 3x \quad\;\; + 5z &= 23 \\ 4x + \;\; y + \;\; z &= 1 \end{aligned}$$

$$A_2 = \begin{bmatrix} 2 & -11 & 2 \\ 3 & 23 & 5 \\ 4 & 1 & 1 \end{bmatrix}$$

$|A| = 47$

$|A_2| = -329$

$$|A_1| = \begin{vmatrix} -11 & 3 & 2 \\ 23 & 0 & 5 \\ 1 & 1 & 1 \end{vmatrix}$$ expand along 2nd column

$$= -3 \begin{vmatrix} 23 & 5 \\ 1 & 1 \end{vmatrix} - 1 \begin{vmatrix} -11 & 2 \\ 23 & 5 \end{vmatrix}$$

$$= -3(18) - 1(-101)$$

$$= 47$$

$$|A_3| = \begin{vmatrix} 2 & 3 & -11 \\ 3 & 0 & 23 \\ 4 & 1 & 1 \end{vmatrix}$$ expand along 2nd column

$$= -3 \begin{vmatrix} 3 & 23 \\ 4 & 1 \end{vmatrix} - 1 \begin{vmatrix} 2 & -11 \\ 3 & 23 \end{vmatrix}$$

$$= -3(-89) - 1(79)$$

$$= 188$$

$$x = \frac{|A_1|}{|A|} = 1$$

$$y = \frac{|A_2|}{|A|} = -7$$

$$z = \frac{|A_3|}{|A|} = 4$$

Definition: A **cofactor** is the signed determinant in the cofactor expansion that's associated with a matrix entry. It's written C_{ij}.

The sign is given by the checkerboard pattern: $\begin{bmatrix} + & - & + & \dots \\ - & + & - & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}$.

Example: Let $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 5 \end{bmatrix}$.
Calculate the cofactors C_{11}, C_{12} and C_{32}.

$$C_{11} = + \begin{vmatrix} 0 & 3 \\ 1 & 5 \end{vmatrix} = +(-3) = -3$$

$$C_{12} = - \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = -(2) = -2$$

$$C_{32} = - \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2$$

Definition: The **cofactor matrix** is the matrix whose entries are the cofactors of A.

Example: Find the cofactor matrix for $A = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 3 \\ 1 & 1 & 5 \end{bmatrix}$.

$$\text{Cofactor Matrix} = \begin{bmatrix} -3 & -2 & 1 \\ 11 & 4 & -3 \\ -6 & -2 & 2 \end{bmatrix}$$

Definition: The **adjoint of** A is the transpose of the cofactor matrix. It's written $\text{adj}(A)$.

Fact: For an $n \times n$ matrix with $|A| \neq 0$:
$A^{-1} = \frac{1}{|A|}\,\text{adj}(A)$.

Example: Let $A = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 2 \\ 3 & -3 & 4 \end{bmatrix}$. Find A^{-1} using the adjoint formula.

Cofactor Matrix $= \begin{bmatrix} 10 & -2 & -9 \\ -14 & 2 & 12 \\ 2 & 0 & -2 \end{bmatrix}$

transpose

$\text{adj}(A) = \begin{bmatrix} 10 & -14 & 2 \\ -2 & 2 & 0 \\ -9 & 12 & -2 \end{bmatrix}$

$|A| = 2\begin{vmatrix} 1 & 2 \\ -3 & 4 \end{vmatrix} - 2\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} + 2\begin{vmatrix} 2 & 1 \\ 3 & -3 \end{vmatrix}$

(expand along 1st row)

$= 2(10) - 2(2) + 2(-9)$

$= -2$

$A^{-1} = \frac{1}{|A|}\,\text{adj}(A)$

$= -\frac{1}{2}\,\text{adj}(A)$

Example: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Find A^{-1} using the adjoint formula.

$$\text{Cofactor Matrix} = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

transpose

$$\operatorname{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$|A| = ad - bc$$

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

4.3 Eigenvalues and Eigenvectors, $n \times n$ Matrices

Example: Find all the eigenvalues of $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -4 & 3 \\ 0 & 0 & 7 \end{bmatrix}$.

Solve, $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & -4-\lambda & 3 \\ 0 & 0 & 7-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(-4-\lambda)(7-\lambda) = 0$$

because the matrix is upper triangular

$$\lambda = 1, -4, 7$$

Fact: The eigenvalues of an upper triangular, lower triangular or diagonal matrix are the diagonal entries.

Integer Roots Theorem
If a polynomial has integer coefficients and the leading coefficient is 1 then any integer roots divide the constant.

Example: Find all the eigenvalues of $A = \begin{bmatrix} 3 & -1 & 1 \\ 7 & -5 & 1 \\ 6 & -6 & 2 \end{bmatrix}$.

Solve $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ 7 & -5-\lambda & 1 \\ 6 & -6 & 2-\lambda \end{vmatrix} = 0$$

expand along 1st row:

$$(3-\lambda)\begin{vmatrix} -5-\lambda & 1 \\ -6 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 7 & 1 \\ 6 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 7 & -5-\lambda \\ 6 & -6 \end{vmatrix} = 0$$

$$(3-\lambda)\left[(-5-\lambda)(2-\lambda)+6\right] + \left[7(2-\lambda)-6\right]$$
$$+ \left[-42 - 6(-5-\lambda)\right] = 0$$

$$(3-\lambda)\left[\lambda^2+3\lambda-4\right] + \left[8-7\lambda\right]$$
$$+ \left[-12+6\lambda\right] = 0$$

Example Continued...

$$\begin{array}{rrrr} & 3\lambda^2 & +9\lambda & -12 \\ -\lambda^3 & -3\lambda^2 & +4\lambda & \\ & & -7\lambda & +8 \\ + & & 6\lambda & -12 \\ \hline -\lambda^3 & & +12\lambda & -16 \end{array} = 0$$

$$\boxed{\lambda^3 - 12\lambda + 16 = 0}$$

Possible Roots: $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16$

$\lambda = -1$: $(-1)^3 - 12(-1) + 16 = 0$? No

$\lambda = 1$: $1^3 - 12(1) + 16 = 0$? No

$\lambda = -2$: $(-2)^3 - 12(-2) + 16 = 0$? No

$\lambda = 2$: $2^3 - 12(2) + 16 = 0$? YES

$\lambda = 2$ is a root of $\lambda^3 - 12\lambda + 16 = 0$

$\Rightarrow$ $\lambda - 2$ " factor " $\lambda^3 - 12\lambda + 16$

$$\begin{array}{r} \lambda^2 + 2\lambda - 8 \\ (\lambda-2)\overline{\big)\,\lambda^3 + 0\lambda^2 - 12\lambda + 16} \\ -(\lambda^3 - 2\lambda^2) \\ \hline 2\lambda^2 - 12\lambda + 16 \\ -(2\lambda^2 - 4\lambda) \\ \hline -8\lambda + 16 \\ -(-8\lambda + 16) \\ \hline 0 \end{array}$$

$$(\lambda-2)(\lambda^2+2\lambda-8) = 0$$

$$(\lambda-2)(\lambda+4)(\lambda-2) = 0$$

$$(\lambda-2)^2(\lambda+4) = 0$$

$$\lambda = 2, -4 \quad \checkmark$$

or $\lambda = 2, 2, -4$ ✓