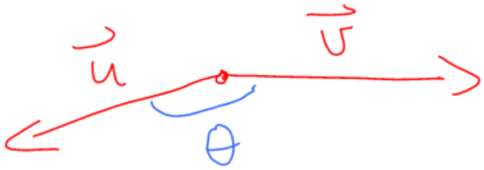
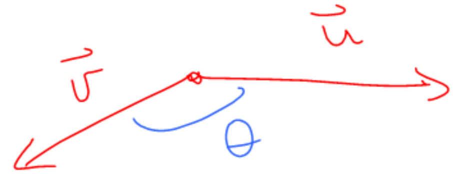
Fact: Let $\vec{u}$ and $\vec{v}$ be in $\mathbb{R}^n$. The angle θ between $\vec{u}$ and $\vec{v}$ is defined to be $0° \leq \theta \leq 180°$





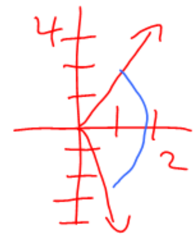
Fact: For all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$: $\quad \vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta$

Comment: In $\mathbb{R}^4$ and higher dimensions, this is a definition of θ.

Comment: In the special case where $\vec{u}$ and $\vec{v}$ are unit vectors, $\vec{u} \cdot \vec{v}$ gives the value of $\cos\theta$.

Example: Find the angle between $\vec{u} = [1, -4]$ and $\vec{v} = [2, 3]$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \, \|\vec{v}\| \cos\theta$$
$$-10 = \sqrt{17}\sqrt{13}\cos\theta$$
$$\frac{-10}{\sqrt{17}\sqrt{13}} = \cos\theta$$
$$\cos^{-1}\left(\frac{-10}{\sqrt{17}\sqrt{13}}\right) = \theta$$
$$\theta \approx 132^\circ$$



Example: If $0° \le \theta < 90°$, what is the sign of $\vec{u} \cdot \vec{v}$?
What if $\theta = 90°$?
What if $90° < \theta \le 180°$?

$0° \le \theta < 90°$	$\theta = 90°$	$90° < \theta \le 180°$
$\Rightarrow \cos\theta > 0$	$\Rightarrow \cos\theta = 0$	$\Rightarrow \cos\theta < 0$
$\Rightarrow \vec{u} \cdot \vec{v} > 0$	$\Rightarrow \vec{u} \cdot \vec{v} = 0$	$\Rightarrow \vec{u} \cdot \vec{v} < 0$

$$\vec{u} \cdot \vec{v} = \underbrace{\|\vec{u}\| \, \|\vec{v}\|}_{>0} \cos\theta$$

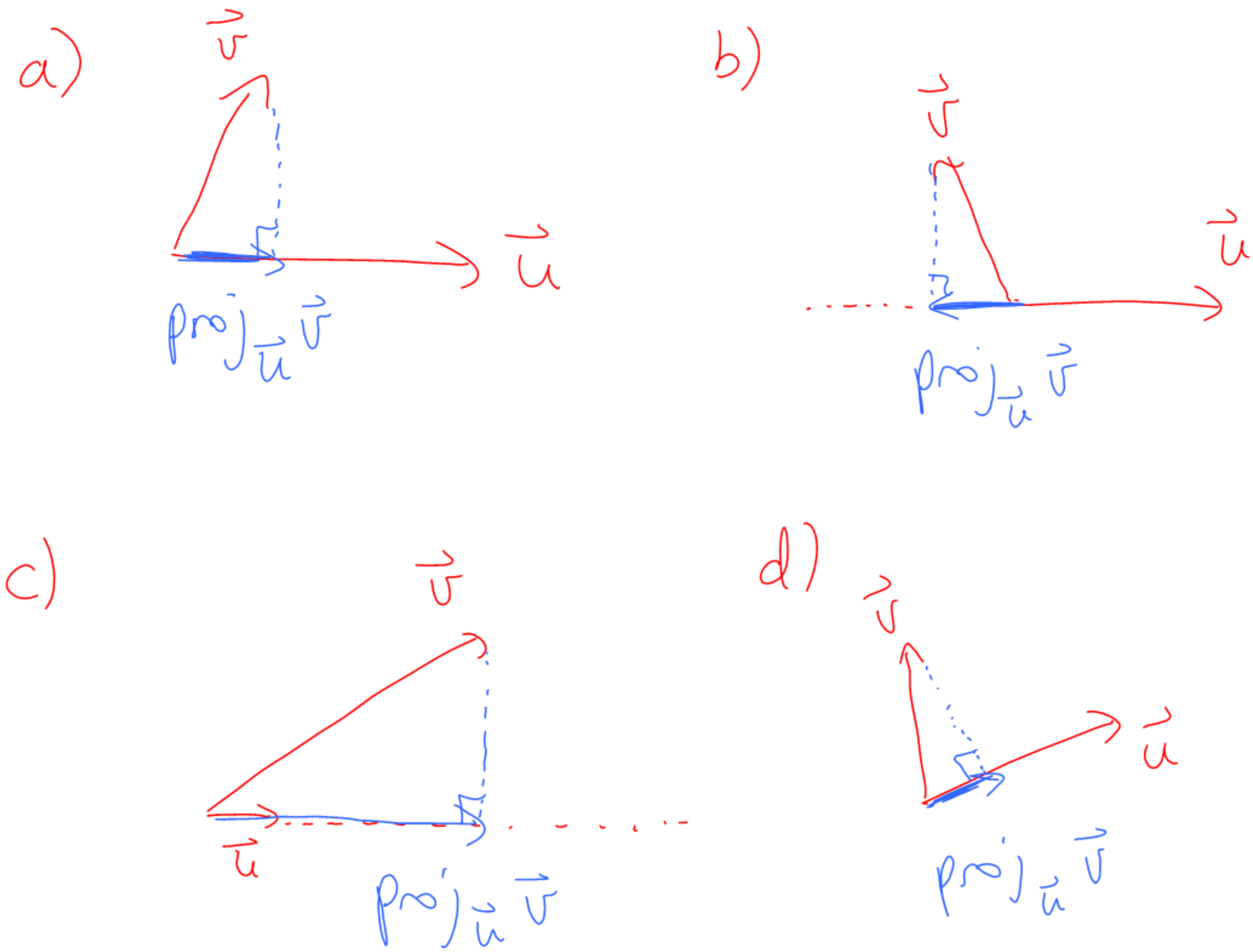
Definition: Vectors $\vec{u}$ and $\vec{v}$ are **orthogonal** if $\vec{u} \cdot \vec{v} = 0$.

Comment: The following statements are equivalent in 2D and 3D:
Vectors $\vec{u}$ and $\vec{v}$ are perpendicular (geometry language)
Vectors $\vec{u}$ and $\vec{v}$ are orthogonal (algebra language)

Comment: In higher dimensions it's more appropriate to use the word **orthogonal** rather than perpendicular.

Definition: The **projection** of $\vec{v}$ onto $\vec{u}$ is written $\text{proj}_{\vec{u}}\vec{v}$. This could be read as the projection onto $\vec{u}$ of $\vec{v}$.

Example: Let's draw a few instances of $\text{proj}_{\vec{u}}\vec{v}$



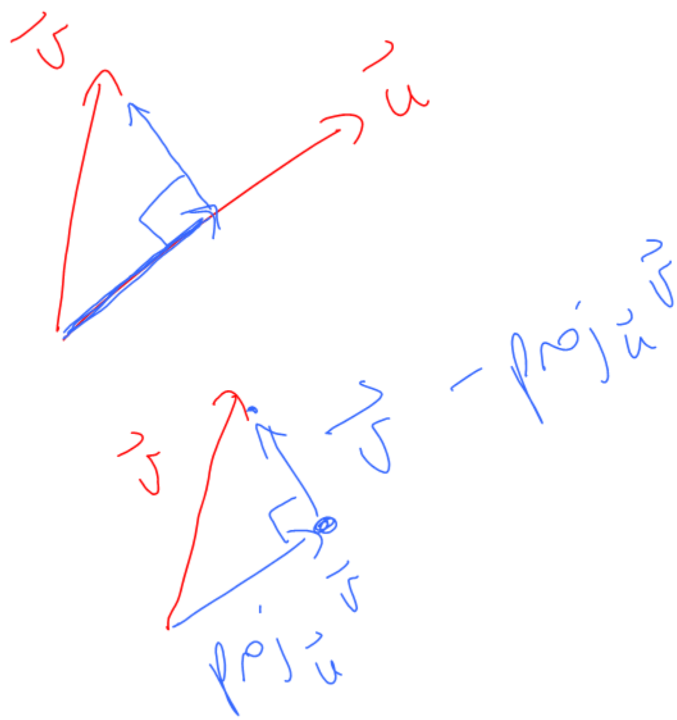
Fact: $\mathrm{proj}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||^2}\vec{u}$

Example: Find $\mathrm{proj}_{\vec{u}}\vec{v}$ for $\vec{u} = [1,2]$ and $\vec{v} = [1,3]$

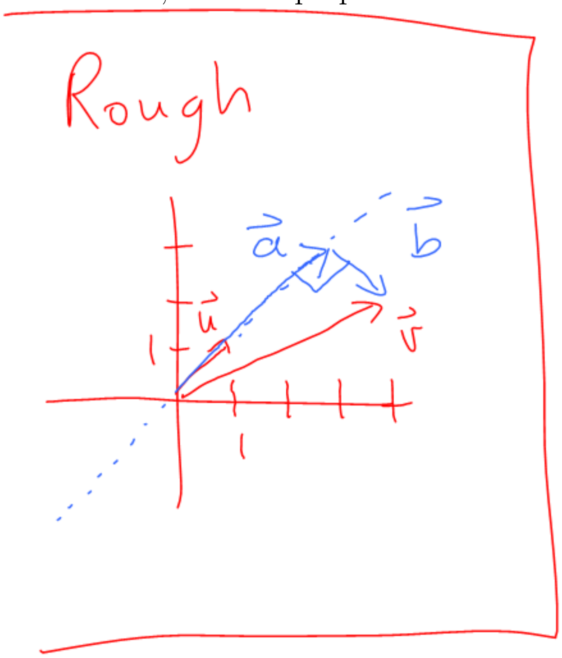
$$\mathrm{proj}_{\vec{u}}\vec{v} = \frac{\vec{u}\cdot\vec{v}}{||\vec{u}||^2}\vec{u}$$

$$= \frac{7}{5}\vec{u} \quad \text{or} \quad \frac{7}{5}[1,2]$$

Fact: Given vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^n$, there is exactly one way to decompose $\vec{v}$ into two vectors that are parallel and perpendicular to $\vec{u}$.



Example: Let $\vec{u} = [1,1]$ and $\vec{v} = [4,2]$. Find vectors $\vec{a}$ and $\vec{b}$ so that $\vec{v} = \vec{a} + \vec{b}$, $\vec{a}$ is parallel to $\vec{u}$, and $\vec{b}$ is perpendicular to $\vec{u}$.



$$\vec{a} = \operatorname{proj}_{\vec{u}} \vec{v} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \vec{u} = \frac{6}{2}[1,1] = 3[1,1] = [3,3]$$

$$\vec{v} = \vec{a} + \vec{b}$$

$$\vec{b} = \vec{v} - \vec{a} = [4,2] - [3,3] = [1,-1]$$