1.2 Length and Angle

Example: Let $\vec{u} = [1, 4, 2, -9]$ and $\vec{v} = [2, 3, -2, -1]$. Calculate the dot product $\vec{u} \cdot \vec{v}$

Example: Calculate:

a) $[1, 5] \cdot [2, -3]$

$= 1(2) + 5(-3)$

$= -13$

b) $[1, 5] \cdot [2, -3, 0]$

undefined

c) $[u_1, u_2] \cdot [u_1, u_2]$

$= u_1^2 + u_2^2$

Fact: Three Properties of the Dot Product

Let $\vec{u}, \vec{v}$ be in $\mathbb{R}^n$. Then:

1) $\vec{u} \cdot \vec{u} \geq 0$
2) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
3) $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = \vec{0}$

Example: Break Property 3 into two statements, and decide which is more obvious.

If $\vec{u} \cdot \vec{u} = 0$ then $\vec{u} = \vec{0}$. (LESS OBVIOUS)

AND

If $\vec{u} = \vec{0}$ then $\vec{u} \cdot \vec{u} = 0$ (MORE OBVIOUS)

Example: Simplify:

a) $(\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$

$= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v}$

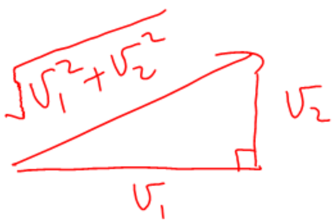
$\vec{u}\,\vec{u}$ is nonsense

b) $3\vec{u}\cdot(-2\vec{v}+5\vec{w})$

$= -6\,\vec{u}\cdot\vec{v} + 15\,\vec{u}\cdot\vec{w}$

Definition: The **length** of $\vec{v}$ is written $||\vec{v}||$. If $\vec{v}=[v_1,v_2,\ldots,v_n]$ then $||\vec{v}|| = \sqrt{v_1^2+v_2^2+\ldots+v_n^2}$.

Example: Draw a picture to show that in 2D this is the Pythagorean Theorem.



Example: Calculate:

a) $||[1,1,1,-2]||$

$= \sqrt{1+1+1+4}$

$= \sqrt{7}$

b) $||[3,-1]||$

$= \sqrt{9+1}$

$= \sqrt{10}$

c) $[3,-1]\cdot[3,-1]$

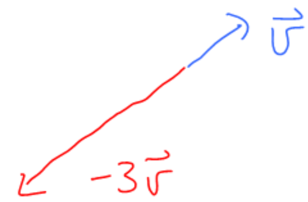
$= 9+1$

$= 10$

Fact: $\vec{v}\cdot\vec{v} = ||\vec{v}||^2$ for all $\vec{v}$

Example: Let $\vec{v} = [v_1, v_2, v_3]$. Simplify $||-3\vec{v}||$.

$$= || [-3v_1, -3v_2, -3v_3] ||$$
$$= \sqrt{9v_1^2 + 9v_2^2 + 9v_3^2}$$
$$= \sqrt{9(v_1^2 + v_2^2 + v_3^2)}$$
$$= \sqrt{9}\sqrt{v_1^2 + v_2^2 + v_3^2}$$
$$= 3\ ||\vec{v}||$$

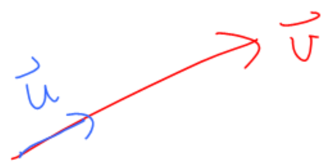


Length of red vector = 3 (length of blue vector)

Fact: $||c\vec{v}|| = |c|\ \ ||\vec{v}||$ for all vectors $\vec{v}$ and real numbers c.

Definition: A **unit vector** is a vector that has length one. **Normalizing** a vector $\vec{v}$ means finding a unit vector in the same direction as $\vec{v}$.

Fact: The following vector has length one and the same direction as $\vec{v}$ (provided that $\vec{v} \neq \vec{0}$):
$\vec{u} = \frac{1}{||\vec{v}||}\vec{v}$



Example: Normalize $\vec{v} = [4, -2, 1]$

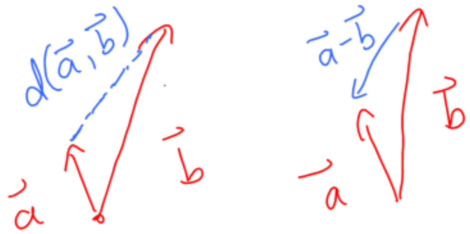
$$||\vec{v}|| = \sqrt{16+4+1}$$
$$= \sqrt{21}$$

$$\vec{u} = \frac{1}{\sqrt{21}}[4, -2, 1]$$

$\vec{u}$ has length 1 and points in the same direction as $\vec{v}$.

Definition: The **distance** between $\vec{a}$ and $\vec{b}$ is written $d(\vec{a}, \vec{b})$. It is calculated by $d(\vec{a}, \vec{b}) = ||\vec{a} - \vec{b}||$

Example: Draw a picture to illustrate the above formula.



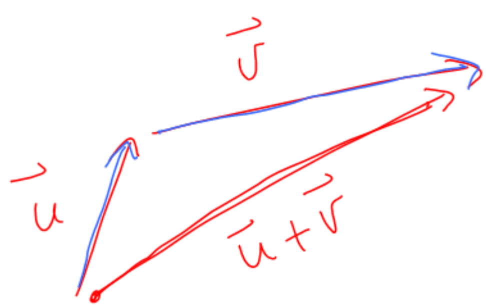
Example: Find the distance between $\vec{a} = [2, -1]$ and $\vec{b} = [3, -6]$

$$\vec{a} - \vec{b} = [-1, 5]$$

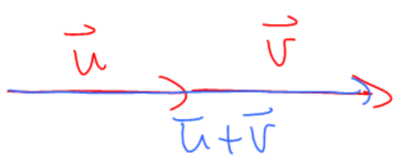
$$||\vec{a} - \vec{b}|| = \sqrt{26}$$

$$d(\vec{a}, \vec{b}) = \sqrt{26}$$

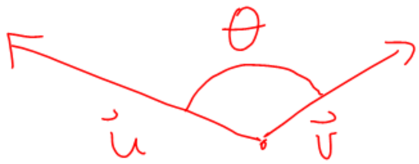
Fact: The Triangle Inequality
For all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$: $||\vec{u} + \vec{v}|| \leq ||\vec{u}|| + ||\vec{v}||$

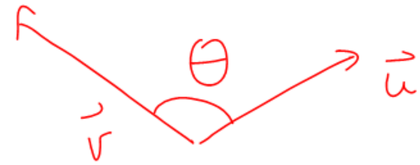


$||\vec{u}+\vec{v}||$ could equal $||\vec{u}|| + ||\vec{v}||$ if $\vec{u}$ and $\vec{v}$ point in the same direction.



Fact: Let $\vec{u}$ and $\vec{v}$ be in $\mathbb{R}^n$. The angle θ between $\vec{u}$ and $\vec{v}$ is defined to be $0° \leq \theta \leq 180°$





Fact: For all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$: $\quad \vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta$

Comment: In $\mathbb{R}^4$ and higher dimensions, this is a definition of θ.

Comment: In the special case where $\vec{u}$ and $\vec{v}$ are unit vectors, $\vec{u} \cdot \vec{v}$ gives the value of $\cos\theta$.