6.2 Series Solutions Cont'd

Ex: Find the first 3 nonzero terms of y_1 and y_2 using $y = \sum_{n=0}^{\infty} c_n x^n$:

$$y'' - (1+x)y = 0$$

$$\vdots$$

$$(2c_2 - c_0) + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} - c_k - c_{k-1}\right]x^k = 0$$

Set coefficients $= 0$

$$2c_2 - c_0 = 0$$

$$(k+2)(k+1)c_{k+2} - c_k - c_{k-1} = 0$$

Recurrence Relation

$$\left[\begin{array}{l} c_2 = \dfrac{c_0}{2} \\ \\ c_{k+2} = \dfrac{c_k + c_{k-1}}{(k+2)(k+1)} \quad \text{for } k \geq 1 \end{array} \right]$$

c_0 : unknown

c_1 : unknown

$c_2 = \dfrac{c_0}{2}$ ↑ in terms of c_0, c_1

$(k=1)$ $\quad C_3 = \dfrac{C_1 + C_0}{3\cdot 2} = \dfrac{C_1}{6} + \dfrac{C_0}{6}$

$(k=2)$ $\quad C_4 = \dfrac{C_2 + C_1}{4\cdot 3} = \dfrac{C_2}{12} + \dfrac{C_1}{12} = \dfrac{C_0}{24} + \dfrac{C_1}{12}$

$$y = C_0 + C_1 x + C_2 x^2 + \dots$$

$$y = C_0 + C_1 x + \frac{C_0}{2} x^2 + \left(\frac{C_1}{6} + \frac{C_0}{6}\right) x^3 + \left(\frac{C_0}{24} + \frac{C_1}{12}\right) x^4 + \dots$$

$$y = C_0 \underbrace{\left[1 + \frac{x^2}{2} + \frac{x^3}{6} + \dots \right]}_{y_1} + C_1 \underbrace{\left[x + \frac{x^3}{6} + \frac{x^4}{12} + \dots \right]}_{y_2}$$

Ex: Find the first 2 nonzero terms of y_1 and y_2 for $y'' - (\sin x) y = 0$

using $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

and $y = C_0 + C_1 x + C_2 x^2 + \dots$

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + \dots$$

$$y' = C_1 + 2C_2 x + 3C_3 x^2 + 4C_4 x^3 + 5C_5 x^4 + \dots$$

$$y'' = 2C_2 + 6C_3 x + 12C_4 x^2 + 20C_5 x^3 + \dots$$

Can't predict # of terms needed.
Can add more terms later if needed.

$$y'' - (\sin x)\, y = 0$$

$$(2C_2 + 6C_3 x + 12C_4 x^2 + 20C_5 x^3 + \dots)$$
$$-\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \dots\right)(C_0 + C_1 x + C_2 x^2 + \dots) = 0$$

$$(2C_2 + 6C_3 x + 12C_4 x^2 + 20C_5 x^3 + \dots)$$
$$+\left(-x + \frac{x^3}{6} - \frac{x^5}{120} + \dots\right)(C_0 + C_1 x + C_2 x^2 + \dots) = 0$$

$$2C_2 + 6C_3 x + 12C_4 x^2 + 20C_5 x^3 + \dots$$
$$-C_0 x \quad -C_1 x^2 \quad -C_2 x^3 - \dots \qquad = 0$$
$$\frac{C_0 x^3}{6} + \dots$$

$$2C_2 + (6C_3 - C_0)x + (12C_4 - C_1)x^2 + \left(20C_5 - C_2 + \frac{C_0}{6}\right)x^3 + \dots = 0$$

Set coefficients $= 0$

$$2C_2 = 0 \implies C_2 = 0$$
$$6C_3 - C_0 = 0 \implies C_3 = \frac{C_0}{6}$$
$$12C_4 - C_1 = 0 \implies C_4 = \frac{C_1}{12}$$

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

$$y = C_0 + C_1 x + \frac{C_0}{6} x^3 + \frac{C_1}{12} x^4 + \cdots$$

$$y = C_0 \underbrace{\left[1 + \frac{x^3}{6} + \cdots \right]}_{y_1} + C_1 \underbrace{\left[x + \frac{x^4}{12} + \cdots \right]}_{y_2}$$

Sigma notation method is more powerful:
Can generate as many coefficients as you need quickly.

7.1 The Laplace Transform

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

$f(t)$: function of t; $F(s)$: function of s

Notation: F means the transformed version of f.

Note: s is restricted to $s > 0$ so that $e^{-st}, te^{-st} \to 0$ as $t \to \infty$.

Ex: Find $\mathcal{L}\{t\}$

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

D	I
$\oplus$ t	e^{-st}
$\ominus$ 1	$-\frac{1}{s}e^{-st}$
0	$\frac{1}{s^2}e^{-st}$

$$= \left[-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right]_{t=0}^{t=\infty}$$

$$= 0 - \left[0 - \frac{1}{s^2}\right]$$

$$= \frac{1}{s^2}$$