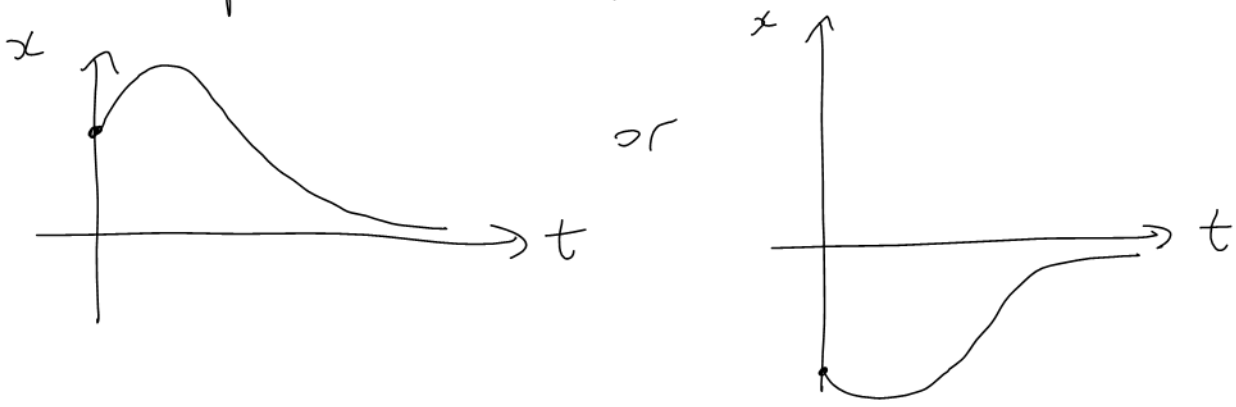
5.1 Spring-Mass Systems Cont'd

Consider damped motion:

$$mx'' + \beta x' + kx = 0$$

3 Scenarios:

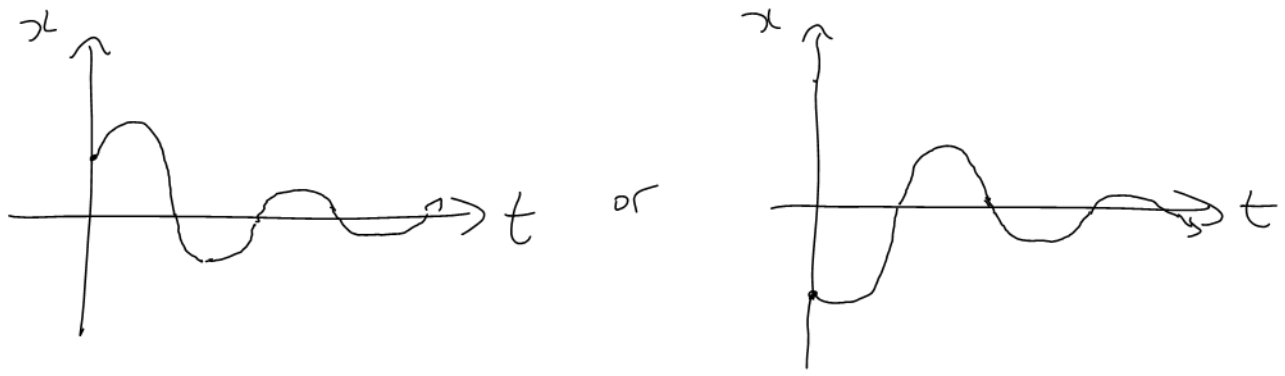
1) Overdamped motion (no oscillation)



$$x = C_1 e^{m_1 t} + C_2 e^{m_2 t}$$

auxiliary equation has distinct real roots

2) Underdamped motion (oscillation)



$$x = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$$

auxiliary equation has complex roots

3) Critically-damped motion
Overdamped motion, but borderline

$$x = e^{m_1 t}(C_1 + C_2 t)$$

auxiliary equation has repeated real roots

Driven Motion

$$mx'' + \beta x' + kx = f(t)$$

$f(t)$ ← external force

Brief Ex:

Solve $x'' + 2x' + 5x = \cos t$

$x(0) = 0$, $x'(0) = 0$

Similar to Section 4.4 but x is a function of t.

1) x_c

$$m^2 + 2m + 5 = 0$$

$$m = -1 \pm 2i$$

$$x_c = e^{-t}(C_1 \cos 2t + C_2 \sin 2t)$$

2) x_p

$$x_p = A\cos t + B\sin t \quad \text{(not bad case)}$$

3) $x_p \to DE$

$\vdots$

$x_p = \frac{1}{5}\cos t + \frac{1}{10}\sin t$

4) $x = x_c + x_p$

$x = e^{-t}(C_1 \cos 2t + C_2 \sin 2t) + \frac{1}{5}\cos t + \frac{1}{10}\sin t$

5) Initial Conditions

$\vdots$

$C_1 = -\frac{1}{5}$

$C_2 = -\frac{3}{20}$

$$x = e^{-t}\left(-\frac{1}{5}\cos 2t - \frac{3}{20}\sin 2t\right) + \frac{1}{5}\cos t + \frac{1}{10}\sin t$$

6.2 Series Solutions

We'll solve $f(x)y'' + g(x)y' + h(x)y = 0$

using $y = \sum_{n=0}^{\infty} c_n x^n$ $\left(y = c_0 + c_1 x + c_2 x^2 + \ldots\right)$

Goal: Find the first few coefficients c_0, c_1, c_2, etc.

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$y' = \sum_{n=1}^{\infty} n C_n x^{n-1}$$

Note: $n \geq 1$
Omit $0 C_0 x^{-1}$

$$y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

Note: $n \geq 2$
Omit $0 C_0 x^{-2}$, $0 C_1 x^{-1}$

Negative exponents will be inconvenient. We can omit these terms because they are zero.

Ex: Solve $y'' - xy = 0$ using $y = \sum_{n=0}^{\infty} C_n x^n$.
Find the first 3 nonzero terms of y_1 and y_2.

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$y' = \sum_{n=1}^{\infty} n C_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

DE: $y'' - xy = 0$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$k = n-2$

$n = k+2$

$n=2 \Rightarrow k=0$

$k = n+1$

$n = k-1$

$n=0 \Rightarrow k=1$

Rewrite using k.
All series start at the biggest k-value ($k=1$)

$$\boxed{k=0 \text{ term}} + \sum_{k=1}^{\infty} \sim - \sum_{k=1}^{\infty} \sim = 0$$

(1^{st} term)

$$2c_2 x^0 + \sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2} x^k - \sum_{k=1}^{\infty} c_{k-1} x^k = 0$$

$$2c_2 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} - c_{k-1}\right] x^k = 0$$

Fact: If $b_0 + b_1 x + b_2 x^2 + \cdots = 0$ for all x

then $b_0 = 0,\ b_1 = 0,\ b_2 = 0, \ldots$

Plan: Set all coefficients $= 0$.

$$2C_2 = 0$$

$$(k+2)(k+1)C_{k+2} - C_{k-1} = 0 \quad \text{for } k \geq 1$$

"Recurrence Relation"

$$C_2 = 0$$

$$C_{k+2} = \frac{C_{k-1}}{(k+2)(k+1)} \quad \text{for } k \geq 1$$