Assignment 3
is on website
Due Thurs March 12, 8:30am

5.1 Spring-Mass Systems

Ex: A 20 kg mass is attached to a spring with spring constant $\frac{20}{3}\ \frac{N}{m}$. Mass is released from 1m below equilibrium position with velocity 2 $\frac{m}{s}$ upwards.

a) Find $x(t)$

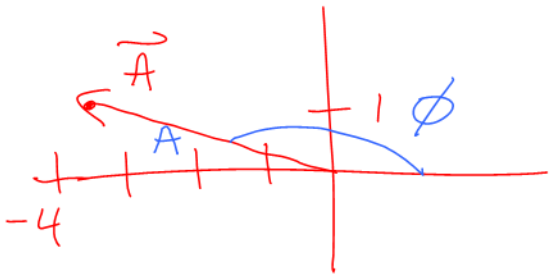
$$x(t) = \cos\frac{t}{\sqrt{3}} - 2\sqrt{3}\sin\frac{t}{\sqrt{3}}$$

b) Write $x(t)$ as $A\sin\left(\frac{t}{\sqrt{3}} + \phi\right)$

Shortcut: $A\cos\phi$ is the coefficient of the sine term.
$A\sin\phi$ " " cosine "

$$A\cos\phi = -2\sqrt{3}$$
$$A\sin\phi = 1$$

$$\vec{A} = [-2\sqrt{3},\ 1\ ]$$



$$A = \sqrt{(-2\sqrt{3})^2 + 1^2}$$

$$= \sqrt{13}$$

$$\phi = \tan^{-1}\left(\frac{1}{-2\sqrt{3}}\right) \quad (+\pi?)$$

$$= \tan^{-1}\left(\frac{1}{-2\sqrt{3}}\right) + \pi$$

$$\approx 2.86 \text{ rads}$$

$$x(t) = A \sin\left(\frac{t}{\sqrt{3}} + \phi\right)$$

$$x(t) = \sqrt{13} \sin\left(\frac{t}{\sqrt{3}} + 2.86\right)$$

c) Mass's maximum displacement from equilibrium position?

$$\pm\sqrt{13} \text{ m}$$

because $\sqrt{13}\sin(\theta)$ ranges from $-\sqrt{13}$ to $\sqrt{13}$

d) Period of motion?

$$x = \sqrt{13} \sin\left(\frac{t}{\sqrt{3}} + 2.86\right)$$

$$\text{Period} = \frac{2\pi}{\left(\frac{1}{\sqrt{3}}\right)} = 2\pi\sqrt{3} \quad \frac{\text{seconds}}{\text{cycle}}$$

e) # of complete cycles after 60 seconds?

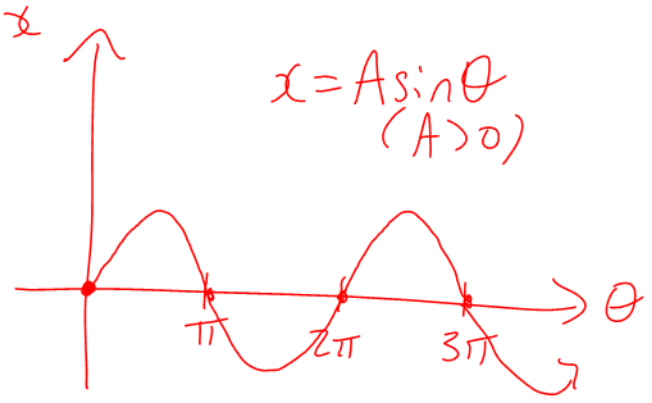
$$\frac{60\ s}{\left(2\pi\sqrt{3}\ \frac{s}{cycle}\right)} \approx 5.5\ cycles$$

5 complete cycles

f) time when the mass first returns to equilibrium position?

$$x = \sqrt{13}\sin\left(\frac{t}{\sqrt{3}} + 2.86\right)$$

$x=0$: $\quad 0 = \sqrt{13}\sin\left(\frac{t}{\sqrt{3}} + 2.86\right)$



$x=0 \Rightarrow \theta = 0, \pi, 2\pi, \ldots$

$\frac{t}{\sqrt{3}} + 2.86 = 0$ nonsense

$\frac{t}{\sqrt{3}} + 2.86 = \pi$

$\Rightarrow t \approx 0.49\ s$

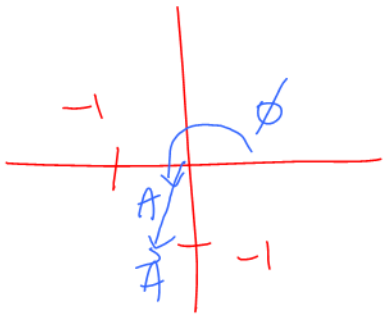
Ex: Suppose $x = e^{-2t}(-\cos 4t - \frac{1}{2}\sin 4t)$. When does the mass first pass through the equilibrium position, heading upwards?

$$-\cos 4t - \frac{1}{2}\sin 4t = A\sin(4t + \phi)$$

$$A\cos\phi = -\frac{1}{2}$$

$$A\sin\phi = -1$$

$$\vec{A} = \left[-\frac{1}{2}, -1\right]$$



$$A = \sqrt{\left(-\frac{1}{2}\right)^2 + 1^2}$$

$$= \sqrt{\frac{5}{4}}$$

$$= \frac{\sqrt{5}}{2}$$

$$\phi = \tan^{-1}\left(\frac{-1}{\left(-\frac{1}{2}\right)}\right) \quad (+\pi?)$$

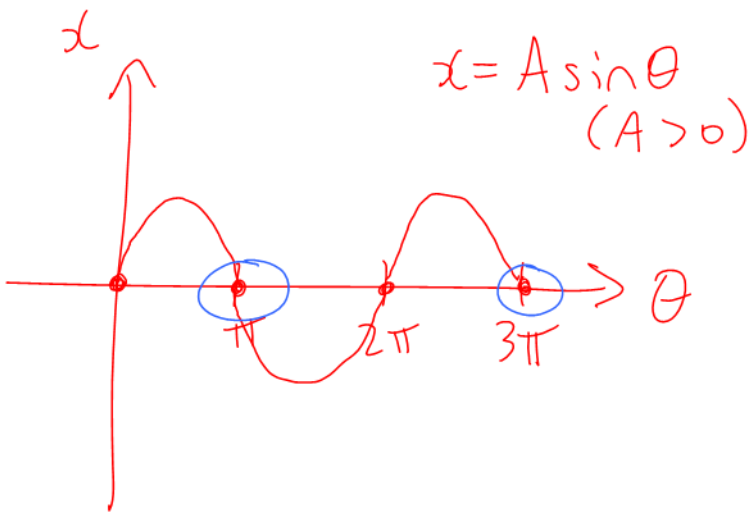
$$= \tan^{-1}(2) + \pi$$

≈ 4.25 rads

$$x = e^{-2t}\left[A\sin(4t+\phi) \right]$$

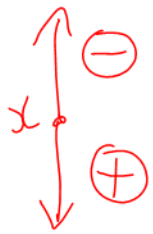
$$x = e^{-2t}\,\frac{\sqrt{5}}{2}\sin(4t+4.25)$$

$$x = \frac{\sqrt{5}}{2}e^{-2t}\sin(4t+4.25)$$



Want $x=0$, becoming negative

$\Rightarrow \theta = \pi, 3\pi, 5\pi, \ldots$



$4t + 4.25 = \pi$ nonsense

$4t + 4.25 = 3\pi$

$4t = 3\pi - 4.25$

$t = \dfrac{3\pi - 4.25}{4} \approx 1.29$ s

Damped Motion

$$mx'' + \beta x' + kx = 0$$

β: fluid resistance is proportional to velocity

Ex: Given $m = 20$ kg and $k = \frac{20}{3}\ \frac{N}{m}$. There is a damping force of magnitude 25 times velocity. Mass is initially at equilibrium, heading downwards at 4 $\frac{m}{s}$. Find $x(t)$.

mx'': $m = 20$ kg

$\beta x'$: $\beta = 25\ \frac{N}{(\frac{m}{s})}$

kx: $k = \frac{20}{3}\ \frac{N}{m}$

$x(0) = 0$
$x'(0) = 4$

(−) / (+)

$$20x'' + 25x' + \frac{20}{3}x = 0$$

Multiply by 3: $60x'' + 75x' + 20x = 0$

Divide by 5: $12x'' + 15x' + 4x = 0$

auxiliary equation: $12m^2 + 15m + 4 = 0$

$$m = \frac{-15 \pm \sqrt{33}}{24}$$

~~$y = C_1 e^{m_1 x} + C_2 e^{m_2 x}$~~

$$x = C_1 e^{m_1 t} + C_2 e^{m_2 t}$$

$$\boxed{x = C_1 e^{\frac{-15+\sqrt{33}}{24} t} + C_2 e^{\frac{-15-\sqrt{33}}{24} t}}$$

$x=0$, $t=0$:

$$0 = C_1 + C_2 \quad (1)$$

$$x' = \left(\frac{-15+\sqrt{33}}{24}\right) C_1 e^{\frac{-15+\sqrt{33}}{24} t} + \left(\frac{-15-\sqrt{33}}{24}\right) C_2 e^{\frac{-15-\sqrt{33}}{24} t}$$

$x'=4$, $t=0$:

$$4 = \left(\frac{-15+\sqrt{33}}{24}\right) C_1 + \left(\frac{-15-\sqrt{33}}{24}\right) C_2 \quad (2)$$

(1): $C_1 = -C_2$

$C_1 = -C_2 \rightarrow (2)$:

$$4 = \left(\frac{-15+\sqrt{33}}{24}\right)(-C_2) + \left(\frac{-15-\sqrt{33}}{24}\right) C_2$$

$$4 = -\frac{\sqrt{33}}{24} C_2 - \frac{\sqrt{33}}{24} C_2$$

$$4 = -\frac{2\sqrt{33}}{24} C_2$$

$$-\frac{48}{\sqrt{33}} = C_2 \Rightarrow C_1 = \frac{48}{\sqrt{33}}$$

$$x = \frac{48}{\sqrt{33}} e^{\frac{-15+\sqrt{33}}{24}t} - \frac{48}{\sqrt{33}} e^{\frac{-15-\sqrt{33}}{24}t}$$