Test Review

Ex: Set up a DE but don't solve.

A bacteria population naturally grows at a rate proportional to the square root of the population. After being sprayed with soap, 3% of the population dies each second as the natural growth continues.

let P = population (# of bacteria)
t = time (seconds)

$$\frac{dP}{dt} = k\sqrt{P} - 0.03P \qquad (k: \text{constant})$$

Ex: Solve $y'' - 2y' - 8y = 9e^{4x} - 7e^{3x}$

1) y_c

$$m^2 - 2m - 8 = 0$$
$$(m-4)(m+2) = 0$$
$$m = 4, -2$$
$$y_1 = e^{4x}, \quad y_2 = e^{-2x}$$
$$y_c = C_1 e^{4x} + C_2 e^{-2x}$$

2) y_p

$y_p = \cancel{Ae^{4x}} + Be^{3x}$?

bad case

$y_p = Axe^{4x} + Be^{3x}$

3) $y_p \to DE$ (find A, B)

$y_p = Axe^{4x} + Be^{3x}$

$y_p' = 4Axe^{4x} + Ae^{4x} + 3Be^{3x}$

$y_p'' = (16Axe^{4x} + 4Ae^{4x}) + 4Ae^{4x} + 9Be^{3x}$

$= 16Axe^{4x} + 8Ae^{4x} + 9Be^{3x}$

DE: $y'' - 2y' - 8y = 9e^{4x} - 7e^{3x}$

$(16Axe^{4x} + 8Ae^{4x} + 9Be^{3x}) - 2(4Axe^{4x} + Ae^{4x} + 3Be^{3x})$

$-8(Axe^{4x} + Be^{3x}) = 9e^{4x} - 7e^{3x}$

$\underbrace{(16A - 8A - 8A)}_{0}xe^{4x} + \underbrace{(8A - 2A)}_{6A}e^{4x} + \underbrace{(9B - 6B - 8B)}_{-5B}e^{3x}$

$= 0xe^{4x} + 9e^{4x} - 7e^{3x}$

Match Coefficients:

$$6A = 9 \Rightarrow A = \frac{9}{6} = \frac{3}{2}$$

$$-5B = -7 \Rightarrow B = \frac{7}{5}$$

$$y_p = Axe^{4x} + Be^{3x}$$

$$y_p = \frac{3}{2}xe^{4x} + \frac{7}{5}e^{3x}$$

4) y

$$y = y_c + y_p$$

$$y = C_1e^{4x} + C_2e^{-2x} + \frac{3}{2}xe^{4x} + \frac{7}{5}e^{3x}$$

5) ~~Initial Conditions~~

Ex: Find a second solution given that $y_1 = e^{7x}$ is a solution:

$$y'' - \left(\frac{1}{x} + 7\right)y' + \frac{7}{x}y = 0$$

DE is in standard form ✓

$$P(x) = -\left(\frac{1}{x} + 7\right)$$

$$-\int P(x)dx = \int \left(\frac{1}{x} + 7\right)dx$$

$$= \ln|x| + 7x$$

$$e^{-\int P(x)dx} = e^{\ln|x| + 7x}$$

$$= e^{\ln|x|} \cdot e^{7x}$$

$$= |x| e^{7x}$$

$$= x e^{7x} \quad (x>0)$$

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{y_1^2} dx$$

$$= e^{7x} \int \frac{x e^{7x}}{(e^{7x})^2} dx$$

$$= e^{7x} \int \frac{x e^{7x}}{e^{14x}} dx$$

$$= e^{7x} \int x e^{-7x} dx$$

	D	I
⊕	x	e^{-7x}
⊖	1	$\frac{-1}{7} e^{-7x}$
	0	$\frac{1}{49} e^{-7x}$

$$= e^{7x} \left[\frac{-x}{7} e^{-7x} - \frac{1}{49} e^{-7x} \right]$$

$= -\frac{x}{7} - \frac{1}{49}$ ✓

Any nonzero multiple of this is also correct.

e.g. $y_2 = \frac{x}{7} + \frac{1}{49}$ ✓

$y_2 = 7x + 1$ ✓

Ex: Solve $y'' + 6y' + 9y = 0$,

$y(1) = 6, \quad y'(1) = 9.$

$m^2 + 6m + 9 = 0$

$(m+3)^2 = 0$

$m = -3, -3$

$y_1 = e^{-3x}, \quad y_2 = xe^{-3x}$

$$\boxed{y = C_1 e^{-3x} + C_2 x e^{-3x}}$$

$y=6$, $x=1$: $\quad 6 = C_1 e^{-3} + C_2 e^{-3}$

$6e^3 = C_1 + C_2$ (1)

$y' = -3C_1 e^{-3x} + C_2(-3xe^{-3x} + e^{-3x})$

$y'=9$, $x=1$: $\quad 9 = -3C_1 e^{-3} + C_2(-3e^{-3} + e^{-3})$

$$9e^3 = -3C_1 + C_2(-3+1)$$

$$9e^3 = -3C_1 - 2C_2 \qquad (2)$$

$3 \times (1)$: $\quad 3C_1 + 3C_2 = 18e^3$

(2): $\quad -3C_1 - 2C_2 = 9e^3$

$+$

$$C_2 = 27e^3$$

$C_2 = 27e^3 \rightarrow (1)$: $\quad C_1 + 27e^3 = 6e^3$

$$C_1 = -21e^3$$

$$y = C_1 e^{-3x} + C_2 x e^{-3x}$$

$$y = -21e^3 e^{-3x} + 27e^3 x e^{-3x}$$

$$y = -21e^{3-3x} + 27x e^{3-3x}$$