4.7 Cauchy-Euler DEs Cont'd

To solve nonhomogeneous Cauchy-Euler DEs, We'll use Variation of Parameters.

Ex: Solve $x^2y'' - 6xy' + 12y = \frac{1}{x}$.

1) y_c

$x^2y'' - 6xy' + 12y = 0$

Cauchy-Euler

$m(m-1) - 6m + 12 = 0$

$m^2 - 7m + 12 = 0$

$(m-3)(m-4) = 0$

$m = 3, 4$

$y_1 = x^3, \quad y_2 = x^4$

$y_c = C_1x^3 + C_2x^4$

2) W, W_1, W_2

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} x^3 & x^4 \\ 3x^2 & 4x^3 \end{vmatrix}$$

$$= 4x^6 - 3x^6$$

$$= x^6$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x^4 \\ x^{-3} & \sim \end{vmatrix}$$

$$= -x$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$= \begin{vmatrix} x^3 & 0 \\ \sim & x^{-3} \end{vmatrix}$$

$$= 1$$

Caution: DE must be in standard form.

$$y'' - \frac{6y'}{x} + \frac{12y}{x^2} = \frac{1}{x^3}$$

$f(x)$

3) u_1

$$u_1' = \frac{W_1}{W}$$

$$= \frac{-x}{x^6}$$

$$= -x^{-5}$$

$$u_1 = \int -x^{-5}\,dx$$

$$= \frac{x^{-4}}{4}$$

4) u_2

$$u_2' = \frac{W_2}{W}$$

$$= \frac{1}{x^6}$$

$$= x^{-6}$$

$$u_2 = \int x^{-6}\,dx$$

$$= -\frac{1}{5}x^{-5}$$

5) y_p

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \frac{1}{4}x^{-4}x^3 - \frac{1}{5}x^{-5}x^4$$

$$= \left(\frac{1}{4} - \frac{1}{5}\right)x^{-1}$$

$$= \frac{1}{20}x^{-1}$$

6) y

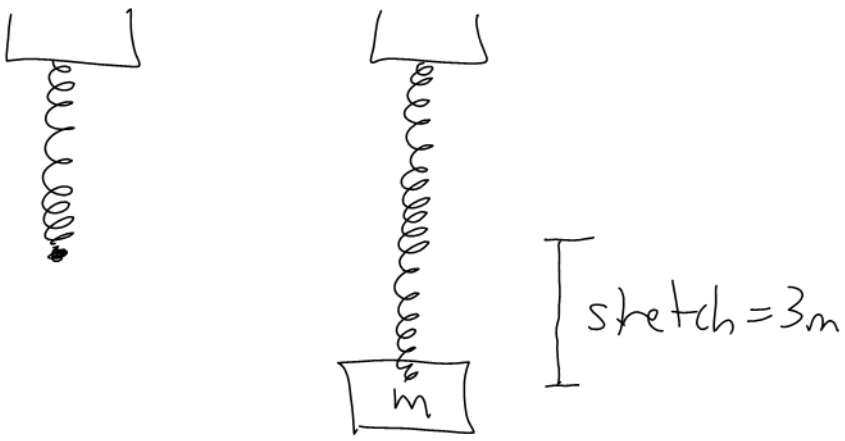
$$y = y_c + y_p$$

$$y = C_1 x^3 + C_2 x^4 + \frac{1}{20x}$$

7) ~~Initial Conditions~~

5.1 Spring-Mass Systems

Ex:



Find the spring constant if $m = 9$ kg.

$g = 9.8 \frac{N}{kg}$

$\Rightarrow W = 9\,kg \times 9.8 \frac{N}{kg}$

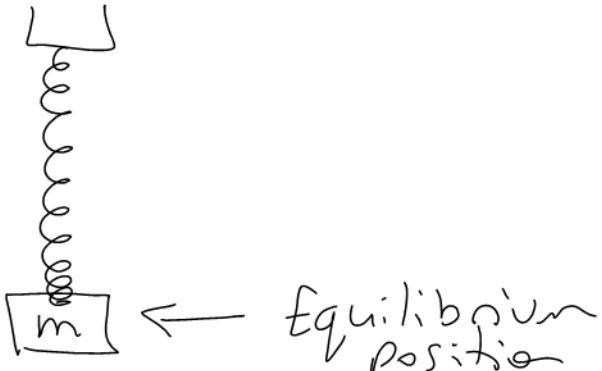
$= 88.2\ N$

$F = kx$ Hooke's Law

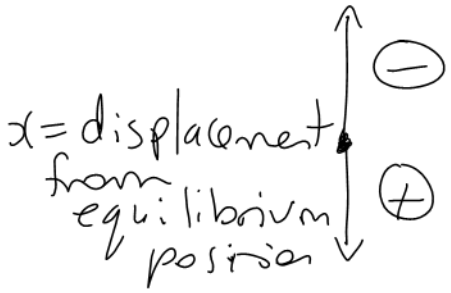
$88.2\ N = k\,(3\ m)$

$29.4 \frac{N}{m} = k$ "spring constant"

Terminology and Notation:



(where mass comes to rest)



$$F_{net} = ma$$

$$-kx = mx''$$

DE: $\boxed{mx'' + kx = 0}$ (assuming no fluid resistance or external force)

More generally: $mx'' + \beta x' + kx = f(t)$

- m: mass, in kg
- β: magnitude of fluid resistance, as a multiple of velocity
- k: spring constant, in $\frac{N}{m}$
- $f(t)$: external force, in N

Ex: A 20 kg mass is attached to a spring with spring constant $\frac{20}{3}$ $\frac{N}{m}$. Mass is released from 1m below equilibrium position with velocity 2 $\frac{m}{s}$ upwards.

a) Find $x(t)$

Periodic Motion: Expect sine and cosine terms.

$$mx'' + kx = 0$$

$$20x'' + \frac{20}{3}x = 0$$

Constant Coefficient DE

auxiliary equation $20m^2 + \frac{20}{3} = 0$

$$20m^2 = -\frac{20}{3}$$

$$m^2 = -\frac{1}{3}$$

$$m = \pm\sqrt{-\frac{1}{3}} \qquad \text{Fi}\sqrt{\frac{1}{3}}$$

$$m = \pm\frac{1}{\sqrt{3}}i \qquad \left(\begin{matrix}\alpha = 0\\ \beta = \frac{1}{\sqrt{3}}\end{matrix}\right)$$

~~$y = e^{\alpha x}(C_1 \cos\beta x + C_2 \sin\beta x)$~~

$$x = e^{\alpha t}(C_1 \cos\beta t + C_2 \sin\beta t)$$

$$\boxed{x = C_1 \cos\frac{t}{\sqrt{3}} + C_2 \sin\frac{t}{\sqrt{3}}}$$

$x(0) = 1$: $\quad 1 = C_1(1)$

$\left(\begin{matrix}x=1\\ t=0\end{matrix}\right)$ $\quad C_1 = 1$

$$\boxed{x = \cos\frac{t}{\sqrt{3}} + C_2 \sin\frac{t}{\sqrt{3}}}$$

$x'(0) = -2$

$\left(\begin{matrix}x'=-2\\ t=0\end{matrix}\right)$: $\quad x' = -\frac{1}{\sqrt{3}}\sin\frac{t}{\sqrt{3}} + \frac{C_2}{\sqrt{3}}\cos\frac{t}{\sqrt{3}}$

$$-2 = 0 + \frac{C_2}{\sqrt{3}}(1)$$

$$-2\sqrt{3} = C_2$$

$$x = \cos\frac{t}{\sqrt{3}} - 2\sqrt{3}\sin\frac{t}{\sqrt{3}}$$

b) Write $x(t)$ as $A\sin\left(\frac{t}{\sqrt{3}} + \phi\right)$.

$$\cos\frac{t}{\sqrt{3}} - 2\sqrt{3}\sin\frac{t}{\sqrt{3}} = A\sin\left(\frac{t}{\sqrt{3}} + \phi\right)$$

Formula Sheet:

$$\sin\left(\frac{t}{\sqrt{3}} + \phi\right) = \sin\frac{t}{\sqrt{3}}\cos\phi + \cos\frac{t}{\sqrt{3}}\sin\phi$$

$$A\sin\left(\frac{t}{\sqrt{3}} + \phi\right) = A\sin\frac{t}{\sqrt{3}}\cos\phi + A\cos\frac{t}{\sqrt{3}}\sin\phi$$

$$\cos\frac{t}{\sqrt{3}} - 2\sqrt{3}\sin\frac{t}{\sqrt{3}} = A\sin\frac{t}{\sqrt{3}}\cos\phi + A\cos\frac{t}{\sqrt{3}}\sin\phi$$

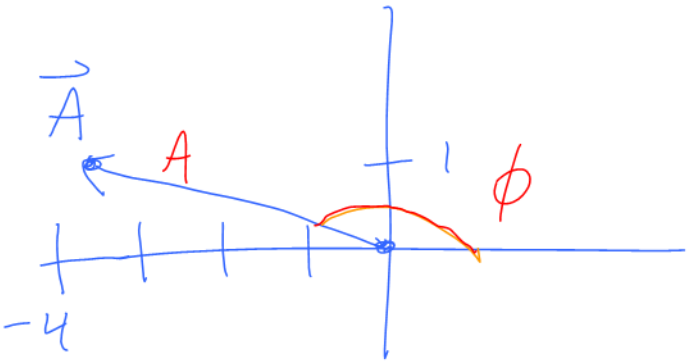
$-2\sqrt{3} = A\cos\phi$

$1 = A\sin\phi$

Shortcut: $A\sin\phi$ is the coefficient of the cosine term.
$A\cos\phi$ " " sine "

$$\begin{cases} x = \cos\frac{t}{\sqrt{3}} - 2\sqrt{3}\sin\frac{t}{\sqrt{3}} \\ A\sin\phi = 1 \quad , \quad A\cos\phi = -2\sqrt{3} \end{cases}$$

$$\vec{A} = [-2\sqrt{3},\ 1]$$



$$A = \sqrt{(-2\sqrt{3})^2 + 1^2}$$

$$= \sqrt{13}$$

$$\phi = \tan^{-1}\left(\frac{1}{-2\sqrt{3}}\right) \quad (+\pi\ ?)$$

$$= \tan^{-1}\left(\frac{1}{(-2\sqrt{3})}\right) + \pi$$

$$\approx 2.86 \text{ rads}$$

$$x(t) = A\sin\left(\frac{t}{\sqrt{3}} + \phi\right)$$

$$x(t) = \sqrt{13}\sin\left(\frac{t}{\sqrt{3}} + 2.86\right)$$