Test 2 (5 Questions)

Fri Feb 27

Covers 3.1, 4.1-4.4

The question from Section 3.1 will be "Set up but don't solve the DE"

4.7 Cauchy-Euler DEs

Consider $x^2y'' - 7xy' + 15y = 0$

Power of x matches the order of the derivative, for each term.

"Cauchy-Euler DE"

Auxiliary equation for Cauchy-Euler DEs:

$x^3y''' \rightarrow m(m-1)(m-2)$

$x^2y'' \rightarrow m(m-1)$

$xy' \rightarrow m$

$y \rightarrow 1$

From above: $m(m-1) - 7m + 15 = 0$

Roots of Auxiliary Equation	Solutions to Cauchy-Euler DE
Distinct real # $m_1, m_2, m_3, \dots$	$x^{m_1}, x^{m_2}, x^{m_3}, \dots$
Repeated real # $m, m, m, \dots$	$x^m, x^m \ln x, x^m (\ln x)^2, \dots$
Complex # $\alpha \pm \beta i$	$x^\alpha [C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)]$

Ex: Solve $x^2 y'' - 7xy' + 15y = 0$

Cauchy-Euler DE

$m(m-1) - 7m + 15 = 0$

$m^2 - m - 7m + 15 = 0$

$m^2 - 8m + 15 = 0$

$(m-3)(m-5) = 0$

$m = 3, 5$

$y_1 = x^3, \quad y_2 = x^5$

$y = C_1 x^3 + C_2 x^5$

Ex: Check that $y = C_1 x^3 + C_2 x^5$ solves $x^2 y'' - 7xy' + 15y = 0$

$$\begin{cases} y = C_1 x^3 + C_2 x^5 \\ y' = 3C_1 x^2 + 5C_2 x^4 \\ y'' = 6C_1 x + 20 C_2 x^3 \end{cases}$$

$$\begin{aligned} LS &= x^2 y'' - 7x y' + 15 y \\ &= x^2 (6C_1 x + 20 C_2 x^3) - 7x (3C_1 x^2 + 5C_2 x^4) \\ &\quad + 15 (C_1 x^3 + C_2 x^5) \\ &= (6 - 21 + 15) C_1 x^3 + (20 - 35 + 15) C_2 x^5 \\ &= 0 C_1 x^3 + 0 C_2 x^5 \\ &= 0 \\ &= RS \end{aligned}$$

this is zero because 5 is a root of the auxiliary equation

	Constant Coefficient DEs (Section 4.3, 4.4, 4.6)	Cauchy-Euler DEs (Section 4.7)
Distinct Real Roots	$e^{m_1 x}, e^{m_2 x}, \ldots$	$x^{m_1}, x^{m_2}, \ldots$
Repeated Real Roots	$e^{m_1 x}, x e^{m_1 x}, x^2 e^{m_1 x}, \ldots$	$x^{m_1}, x^{m_1} \ln x, x^{m_1} (\ln x)^2, \ldots$
Complex Roots	$e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$	$x^{\alpha}(C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x))$

Connection:

$x \to \ln x$

$$e^{m_1 x} \to e^{m_1 \ln x} = e^{\ln x^{m_1}} = x^{m_1}$$

Ex: Solve $x^2 y'' - 3xy' + 4y = 0$,

$y(1) = 6$, $y'(1) = 3$

Cauchy-Euler

$$m(m-1) - 3m + 4 = 0$$

$$m^2 - m - 3m + 4 = 0$$

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0$$

$$m = 2, 2$$

$$y_1 = x^2, \quad y_2 = x^2 \ln x$$

$$\boxed{y = C_1 x^2 + C_2 x^2 \ln x}$$

$y = 6, x = 1$: $\quad 6 = C_1 + 0$

$C_1 = 6$

$$y = 6x^2 + C_2 x^2 \ln x$$

$$y' = 12x + C_2\left[x^2 \cdot \frac{1}{x} + (\ln x)(2x)\right]$$

$y' = 3, x = 1$: $3 = 12 + C_2(1)$

$-9 = C_2$

$$y = 6x^2 - 9x^2 \ln x$$

Ex: Solve $x^2y'' - xy' + 4y = 0$

Cauchy-Euler

$m(m-1) - m + 4 = 0$

$m^2 - m - m + 4 = 0$

$m^2 - 2m + 4 = 0$

$$m = \frac{2 \pm \sqrt{-12}}{2}$$

$\sqrt{4}\sqrt{3}\sqrt{-1}$

$$m = \frac{2 \pm 2\sqrt{3}\,i}{2}$$

$$m = 1 \pm \sqrt{3}\,i \qquad (\alpha = 1,\ \beta = \sqrt{3})$$

$$y = x^{\alpha}\left[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)\right]$$

$$y = x\left[C_1 \cos(\sqrt{3} \ln x) + C_2 \sin(\sqrt{3} \ln x)\right]$$

Ex: Solve $x^3 y''' + 3x^2 y'' + 3xy' - 3y = 0$

Cauchy-Euler

$$m(m-1)(m-2) + 3m(m-1) + 3m - 3 = 0$$

$$m(m^2 - 3m + 2) + 3m^2 - \cancel{3m + 3m} - 3 = 0$$

$$m^3 - \cancel{3m^2} + 2m + \cancel{3m^2} - 3 = 0$$

$$m^3 + 2m - 3 = 0$$

Rational Roots Theorem

p: $\pm 1, \pm 3$ (divisors of the constant)

q: ± 1 (divisors of leading coefficient)

$\frac{p}{q}$: $\pm 1, \pm 3$ possible roots

$m = -1$: $(-1)^3 + 2(-1) - 3 = 0$? No

$m = 1$: $1^3 + 2(1) - 3 = 0$? YES

$m = 1$ is a root of $m^3 + 2m - 3 = 0$

$\Rightarrow$ $(m-1)$ is a factor of $m^3 + 2m - 3$

Long Division

$$
\begin{array}{r}
m^2 + m + 3 \\
(m-1)\overline{\big)\, m^3 + 0m^2 + 2m - 3} \\
-(m^3 - m^2) \\
\hline
m^2 + 2m - 3 \\
-(m^2 - m) \\
\hline
3m - 3 \\
-(3m - 3) \\
\hline
0
\end{array}
$$

$$m^3 + 2m - 3 = 0$$

$$(m-1)(m^2 + m + 3) = 0$$

$$m = 1 \qquad m = \frac{-1 \pm \sqrt{-11}}{2}$$

$$m = \frac{-1}{2} \pm \frac{\sqrt{11}}{2} i \qquad \left(\begin{array}{l} \alpha = \frac{-1}{2} \\ \beta = \frac{\sqrt{11}}{2} \end{array} \right)$$

$$y = x^{-\frac{1}{2}} \left[C_1 \cos\left(\frac{\sqrt{11}}{2} \ln x\right) + C_2 \sin\left(\frac{\sqrt{11}}{2} \ln x\right) \right] + C_3 x$$