4.6 Variation of Parameters

Find a particular solution y_p to

$y'' + ay' + by = f(x)$ when

standard form

$f(x)$, $f'(x)$, $f''(x)$ etc. have infinitely-many different like terms.

Idea: Find solutions y_1, y_2 to $y'' + ay' + by = 0$.

Let $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, $W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$, $W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$

Let $u_1' = \frac{W_1}{W}$ and $u_2' = \frac{W_2}{W}$.

Integrate these to get u_1 and u_2.

Then $y_p = u_1 y_1 + u_2 y_2$.

Ex: Solve $5y'' + 20y = 5\csc 2x$

$y'' + 4y = \csc 2x$ standard form

$f(x)$

1) y_c

$$y'' + 4y = 0$$

$$m^2 + 4 = 0$$

$$m^2 = -4$$

$$m = \pm\sqrt{-4} \quad \sqrt{4}\sqrt{-1}$$

$$m = \pm 2i$$

$y_1 = \cos 2x$, $y_2 = \sin 2x$

$$y_c = C_1 \cos 2x + C_2 \sin 2x$$

2) W, W_1, W_2

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix}$$

$$= 2\cos^2 2x + 2\sin^2 2x$$

$$= 2(\cos^2 2x + \sin^2 2x)$$

$$= 2$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

Caution: Use standard form of DE

$$= \begin{vmatrix} 0 & \sin 2x \\ \csc 2x & 2\cos 2x \end{vmatrix}$$

$$= -(\sin 2x)(\csc 2x)$$

$$= -1$$

$$W_2 = \begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \csc 2x \end{vmatrix}$$

Caution: Use standard form of DE $f(x) = \csc 2x$

$$= (\cos 2x)(\csc 2x)$$

$$= \frac{\cos 2x}{\sin 2x}$$

$$= \cot 2x$$

3) Get u_1

$$u_1' = \frac{W_1}{W}$$

$$= \frac{-1}{2}$$

$$u_1 = \int -\frac{1}{2}\, dx$$

$$u_1 = \frac{-x}{2}$$

Don't use a constant here.

4) Get u_2

$$u_2' = \frac{W_2}{W}$$

$$= \frac{1}{2}\cot 2x$$

$$u_2 = \int \frac{1}{2}\cot 2x \, dx$$

$$= \frac{1}{4}\ln|\sin 2x|$$

Don't use a constant here.

5) Get y_p

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p = -\frac{x}{2}\cos 2x + \frac{1}{4}\ln|\sin 2x|\,(\sin 2x)$$

6) General Solution

$$y = y_c + y_p$$

$$y = C_1\cos 2x + C_2\sin 2x - \frac{x}{2}\cos 2x + \frac{1}{4}\ln|\sin 2x|\,(\sin 2x)$$

~~7) Initial Conditions~~

Ex: Solve $3y'' + 15y' + 18y = 3\cos e^x$

Standard Form $y'' + 5y' + 6y = \underbrace{\cos e^x}_{f(x)}$

1) y_c

$$y'' + 5y' + 6y = 0$$

$$m^2 + 5m + 6 = 0$$

$$(m+2)(m+3) = 0$$

$$m = -2, -3$$

$$y_1 = e^{-2x}, \quad y_2 = e^{-3x}$$

$$y_c = C_1 e^{-2x} + C_2 e^{-3x}$$

2) W, W_1, W_2

$$W = \begin{vmatrix} e^{-2x} & e^{-3x} \\ -2e^{-2x} & -3e^{-3x} \end{vmatrix}$$

$$= -3e^{-5x} + 2e^{-5x}$$

$$= -e^{-5x}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

Caution: Use standard form of DE

$$= \begin{vmatrix} 0 & e^{-3x} \\ \csc e^x & \sim \end{vmatrix}$$

$$= -e^{-3x} \csc e^x$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$= \begin{vmatrix} e^{-2x} & 0 \\ \sim & \cos e^x \end{vmatrix}$$

$$= e^{-2x} \cos e^x$$

3) Find u_1

$$u_1' = \frac{W_1}{W}$$

$$= \frac{-e^{-3x} \cos e^x}{-e^{-5x}}$$

$$= e^{2x} \cos e^x$$

$$u_1 = \int \frac{e^{2x}}{e^x \cdot e^x} \cos e^x \, dx$$

Sub $u = e^x$

$du = e^x dx$

$$= \int u \cos u \, du$$

	D	I
⊕	u	$\cos u$
⊖	1	$\sin u$
	0	$-\cos u$

$$= u \sin u + \cos u$$

$$= e^x \sin e^x + \cos e^x$$

(Don't we a constant here.)

4) Find u_2

$$u_2' = \frac{W_2}{W}$$

$$= \frac{e^{-2x}\cos e^x}{-e^{-5x}}$$

$$= -e^{3x}\cos e^x$$

$$u_2 = \int -\frac{e^{3x}}{e^{2x}\cdot e^x}\cos e^x \, dx$$

Sub $u = e^x$

$du = e^x dx$

$$= \int -u^2 \cos u \, du$$

	D	I
⊕	$-u^2$	$\cos u$
⊖	$-2u$	$\sin u$
⊕	-2	$-\cos u$
	0	$-\sin u$

$$= -u^2 \sin u - 2u\cos u + 2\sin u$$

$$= -e^{2x}\sin e^x - 2e^x \cos e^x + 2\sin e^x$$

5) $y_p = u_1 y_1 + u_2 y_2$

6) $y = y_c + y_p$

7) ~~Initial Conditions~~

To Be Continued