4.3 Homogeneous DEs with Constant Coefficients Cont'd

Rational Roots Theorem

Any rational roots of a polynomial are of the form $\frac{p}{q}$, where p divides the constant and q divides the leading coefficient.

Ex: Solve $2y''' - 7y'' + 7y' - 2y = 0$,
$y(0) = 3$, $y'(0) = 5$, $y''(0) = 9$.

$$2m^3 - 7m^2 + 7m - 2 = 0$$

Rational Roots Theorem

Possibilities for p (divisors of -2): $\pm 1, \pm 2$

Possibilities for q (divisors of 2): $\pm 1, \pm 2$

Possibilities for $\frac{p}{q}$: $\pm 1, \pm \frac{1}{2}, \pm 2$

Possible roots of equation

Sub $m = -1$: $2(-1)^3 - 7(-1)^2 + 7(-1) - 2 = 0$? No

Sub $m=1$: $2(1)^3 - 7(1)^2 + 7(1) - 2 = 0$? YES

$m=1$ is a root of $2m^3 - 7m^2 + 7m - 2$

$\Rightarrow (m-1)$ is a factor of "

Long Division

$$
\begin{array}{r}
2m^2 - 5m + 2 \\
(m-1)\overline{)\,2m^3 - 7m^2 + 7m - 2} \\
-(2m^3 - 2m^2) \\
\hline
-5m^2 + 7m - 2 \\
-(-5m^2 + 5m) \\
\hline
2m - 2 \\
-(2m - 2) \\
\hline
0
\end{array}
$$

$$2m^3 - 7m^2 + 7m - 2 = 0$$

$$(m-1)(2m^2 - 5m + 2) = 0$$

$\swarrow$ $\qquad$ $\downarrow$

$m=1$ $\qquad$ $m = \dfrac{5 \pm \sqrt{9}}{4}$ 3

$$m = 2, \frac{1}{2}$$

$$y_1 = e^x, \quad y_2 = e^{2x}, \quad y_3 = e^{x/2}$$

$$\boxed{y = C_1 e^{x} + C_2 e^{2x} + C_3 e^{x/2}}$$

$y=3$, $x=0$: $3 = C_1 + C_2 + C_3$ ①

$$y' = C_1 e^{x} + 2C_2 e^{2x} + \frac{C_3}{2} e^{x/2}$$

$y'=5$, $x=0$: $5 = C_1 + 2C_2 + \frac{C_3}{2}$ ②

$$y'' = C_1 e^{x} + 4C_2 e^{2x} + \frac{C_3}{4} e^{x/2}$$

$y''=9$, $x=0$: $9 = C_1 + 4C_2 + \frac{C_3}{4}$ ③

Gauss-Jordan

$$\begin{array}{c} \begin{array}{ccc|c} C_1 & C_2 & C_3 & \# \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 1/2 & 5 \\ 1 & 4 & 1/4 & 9 \end{array}\right] \end{array}$$

$$\begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & -1/2 & 2 \\ 0 & 3 & -3/4 & 6 \end{array}\right]$$

$$\begin{array}{l} R_1 - R_2 \\ \\ R_3 - 3R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 3/2 & 1 \\ 0 & 1 & -1/2 & 2 \\ 0 & 0 & 3/4 & 0 \end{array}\right]$$

$$-\frac{3}{4} + \frac{3}{2} = -\frac{3}{4} + \frac{6}{4}$$

$$R_3 \times \left(\frac{4}{3}\right)\begin{bmatrix} 1 & 0 & 3/2 & | & 1 \\ 0 & 1 & -1/2 & | & 2 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

$$\begin{matrix} R_1 - \frac{3}{2}R_3 \\ R_2 + \frac{1}{2}R_3 \end{matrix}\begin{bmatrix} C_1 & C_2 & C_3 & | & \# \\ 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

$C_1 = 1$

$C_2 = 2$

$C_3 = 0$

$$y = C_1 e^{x} + C_2 e^{2x} + C_3 e^{xh}$$

$$\boxed{y = e^{x} + 2e^{2x}}$$

4.4 Nonhomogeneous DEs with Constant Coefficients

Ex: Solve $y'' - 5y' + 6y = 5x$

1) Solve $y'' - 5y' + 6y = 0$

$$m^2 - 5m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$m = 2, 3$$

$$y_1 = e^{2x}, \quad y_2 = e^{3x}$$

$$y_c = C_1 e^{2x} + C_2 e^{3x}$$ "the complementary solution"

2) Find y_p, "the particular solution"

$$f(x) = 5x$$

FACT

y_p consists of all terms in $f(x), f'(x), f''(x), \ldots$ and y_p has unknown coefficients

$$f(x) = 5x$$
$$f'(x) = 5$$
$$f''(x) = 0 \quad \text{(no new like terms)}$$

$$y_p = Ax + B$$

3) Put y_p into DE (Find A and B)

$$\begin{cases} y_p = Ax + B \\ y_p' = A \\ y_p'' = 0 \end{cases}$$

DE: $y'' - 5y' + 6y = 5x$

$$0 - 5(A) + 6(Ax + B) = 5x$$

$$-5A + 6Ax + 6B = 5x$$

$$(6A)x + (-5A + 6B) = 5x + 0$$

Match Coefficients:

$$6A = 5 \quad (1)$$

$$-5A + 6B = 0 \quad (2)$$

$(1) \Rightarrow A = \frac{5}{6}$

$A = \frac{5}{6} \rightarrow (2)$: $-\frac{25}{6} + 6B = 0$

$$6B = \frac{25}{6}$$

$$B = \frac{25}{36}$$

$$y_p = Ax + B$$

$$y_p = \frac{5}{6}x + \frac{25}{36}$$

4) $y = y_c + y_p$

$$y = C_1 e^{2x} + C_2 e^{3x} + \frac{5}{6}x + \frac{25}{36}$$

5) ~~Initial Conditions~~