4.3 Homogeneous DEs with Constant Coefficients Cont'd

Ex: Solve $y'' + 49y = 0$, $y(0) = -5$, $y'(0) = 21$.

$$m^2 + 49 = 0$$

$$m^2 = -49$$

$$m = \pm\sqrt{-49} \quad \sqrt{49}\sqrt{-1} \quad 7i$$

$$m = 0 \pm 7i$$

$\alpha = 0$
$\beta = 7$

$$y_1 = e^{\alpha x}\cos\beta x = e^{0x}\cos 7x = \cos 7x$$

$$y_2 = e^{\alpha x}\sin\beta x = e^{0x}\sin 7x = \sin 7x$$

$$y = C_1 y_1 + C_2 y_2$$

$$\boxed{y = C_1\cos 7x + C_2\sin 7x}$$

$y = -5$, $x = 0$:

$$-5 = C_1(1) + C_2(0)$$

$$-5 = C_1$$

$$\boxed{y = -5\cos 7x + C_2\sin 7x}$$

$y' = 35\sin 7x + 7C_2 \cos 7x$

$y' = 21$, $x = 0$: $21 = 35(0) + 7C_2(1)$

$21 = 7C_2$

$3 = C_2$

$y = -5\cos 7x + 3\sin 7x$

Ex: Solve $y^{(4)} + 8y'' + 16y = 0$

$m^4 + 8m^2 + 16 = 0$

$(m^2)^2 + 8m^2 + 16 = 0$

$(m^2+4)^2 = 0$

$(m^2+4)(m^2+4) = 0$

$m^2 + 4 = 0$

$m^2 = -4$

$m = \pm\sqrt{-4}$ $\sqrt{4}\sqrt{-1}$

$m = \pm 2i$

$m = 0 \pm 2i$

$m = 0 \pm 2i$

$m = 0 \pm 2i$, $0 \pm 2i$

$\alpha = 0$

$\beta = 2$

$y_1 = e^{\alpha x}\cos\beta x = e^{0x}\cos 2x = \cos 2x$

$y_2 = e^{\alpha x}\sin\beta x = \sin 2x$

$y_3 = x\cos 2x$

$y_4 = x\sin 2x$

$$y = C_1 y_1 + C_2 y_2 + C_3 y_3 + C_4 y_4$$

$$y = C_1\cos 2x + C_2\sin 2x + C_3 x\cos 2x + C_4 x\sin 2x$$

or $$y = (C_1 + C_3 x)\cos 2x + (C_2 + C_4 x)\sin 2x$$

Ex: Write the general solution to the DE if:

a) $m = 1, 3, 3$

$y_1 = e^{x}$, $y_2 = e^{3x}$, $y_3 = xe^{3x}$

$$y = C_1 e^{x} + C_2 e^{3x} + C_3 x e^{3x}$$

$$y = C_1 e^{x} + (C_2 + C_3 x)e^{3x}$$

b) $m = 4, 4, -3 \pm 5i$

$y_1 = e^{4x}$, $y_2 = xe^{4x}$,

$y_3 = e^{\alpha x}\cos\beta x = e^{-3x}\cos 5x$

$y_4 = e^{\alpha x}\sin\beta x = e^{-3x}\sin 5x$

$$y = C_1 e^{4x} + C_2 x e^{4x} + C_3 e^{-3x}\cos 5x + C_4 e^{-3x}\sin 5x$$

$$y = (C_1 + C_2 x)e^{4x} + e^{-3x}(C_3\cos 5x + C_4 \sin 5x)$$

c) $m = 7, 7, 7$

$$y_1 = e^{7x},\ y_2 = xe^{7x},\ y_3 = x^2 e^{7x}$$

$$y = C_1 e^{7x} + C_2 x e^{7x} + C_3 x^2 e^{7x}$$

$$y = (C_1 + C_2 x + C_3 x^2)e^{7x}$$

d) $m = 0, 0, 0, 6$

$$y_1 = e^{0x} = 1,\ y_2 = x,\ y_3 = x^2,\ y_4 = e^{6x}$$

$$y = C_1 + C_2 x + C_3 x^2 + C_4 e^{6x}$$

e) $m = 3, 6, 7 \pm 8i$

$$y_1 = e^{3x},\ y_2 = e^{6x},$$

$$y_3 = e^{7x}\cos 8x,\ y_4 = e^{7x}\sin 8x$$

$$y = C_1 e^{3x} + C_2 e^{6x} + e^{7x}(C_3\cos 8x + C_4 \sin 8x)$$

f) $m = 2 \pm 3i,\ 2 \pm 3i$

$$y_1 = e^{2x}\cos 3x$$

$$y_2 = e^{2x}\sin 3x$$

$$y_3 = xe^{2x}\cos 3x$$

$$y_4 = xe^{2x}\sin 3x$$

$$y = C_1y_1 + C_2y_2 + C_3y_3 + C_4y_4$$

$$y = e^{2x}(C_1\cos 3x + C_2\sin 3x) + xe^{2x}(C_3\cos 3x + C_4\sin 3x)$$

ASIDE These are equally correct:

$$y = C_1e^{2x} + C_2e^{3x}$$

$$y = C_1e^{3x} + C_2e^{2x}$$

Rational Roots Theorem

Any rational roots of a polynomial are of the form $\frac{p}{q}$, where p divides the constant and q divides the leading coefficient.

e.g. Solve $2m^3 - 7m^2 + 7m - 2 = 0$

possibilities for p: $\pm 1, \pm 2$

" for q: $\pm 1, \pm 2$

" $\frac{p}{q}$: $\pm 1, \pm \frac{1}{2}, \pm 2$

To Be Continued