4.2 Finding a Second Solution Cont'd

Standard form for a homogeneous 2nd-order linear DE:

$$y'' + P(x)y' + Q(x)y = 0$$

Given one solution y_1, a second solution y_2 is:

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

General solution to the DE:

$$y = C_1 y_1 + C_2 y_2$$

Ex: $y'' + \frac{2}{x}y' - \frac{6}{x^2}y = 0$ has solution $y_1 = x^2$.

a) Find a second linearly independent solution.

Standard Form ✓

$$P(x) = \frac{2}{x}$$

$$\int P(x)\,dx = 2\ln|x|$$

$$-\int P(x)\,dx = -2\ln|x|$$

$$e^{-\int P(x)\,dx} = e^{-2\ln|x|} = e^{\ln|x|^{-2}} = |x|^{-2} = x^{-2}$$

$$y_2 = y_1 \int \frac{e^{-\int p(x)\,dx}}{y_1^2}\,dx$$

$$= x^2 \int \frac{x^{-2}}{(x^2)^2}\,dx$$

$$= x^2 \int \frac{x^{-2}}{x^4}\,x^{-6}\,dx$$

$$= x^2 \left(-\frac{1}{5}x^{-5}\right)$$

$$= -\frac{1}{5}x^{-3}$$

Or, more simply, $y_2 = x^{-3}$

b) General solution to the DE?

$$y = C_1 y_1 + C_2 y_2$$

$$y = C_1 x^2 + C_2 x^{-3}$$

4.3 Homogeneous DEs with Constant Coefficients

DE with Constant Coefficients:

$$y'' - 4y' - 21y = 0$$

Auxiliary equation $m^2 - 4m - 21 = 0$

$(m-7)(m+3) = 0$

$m = 7, -3$

FACT

Roots of Auxiliary Equation	Solutions to DE
Distinct Real Numbers $m_1, m_2, m_3, \ldots$	$e^{m_1 x}, e^{m_2 x}, e^{m_3 x}, \ldots$
Repeated Real Numbers $m_1, m_1, m_1, \ldots$	$e^{m_1 x}, x e^{m_1 x}, x^2 e^{m_1 x}, \ldots$
Complex Numbers $\alpha \pm \beta i$	$e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x)$

Ex: Solve $y'' - 4y' - 21y = 0$.

auxiliary equation $m^2 - 4m - 21 = 0$

$(m-7)(m+3) = 0$

$m = 7, -3$

$y_1 = e^{7x}, \quad y_2 = e^{-3x}$

$y = C_1 y_1 + C_2 y_2 \Rightarrow \boxed{y = C_1 e^{7x} + C_2 e^{-3x}}$

Ex: Check that $y = C_1 e^{7x} + C_2 e^{-3x}$ solves $y'' - 4y' - 21y = 0$.

$$\begin{cases} y = C_1 e^{7x} + C_2 e^{-3x} \\ y' = 7C_1 e^{7x} - 3C_2 e^{-3x} \\ y'' = 49C_1 e^{7x} + 9C_2 e^{-3x} \end{cases}$$

$$LS = (49C_1 e^{7x} + 9C_2 e^{-3x}) - 4(7C_1 e^{7x} - 3C_2 e^{-3x}) - 21(C_1 e^{7x} + C_2 e^{-3x})$$

$$= 49C_1 e^{7x} + 9C_2 e^{-3x} - 28C_1 e^{7x} + 12C_2 e^{-3x} - 21C_1 e^{7x} - 21C_2 e^{-3x}$$

$$= 0C_1 e^{7x} + 0C_2 e^{-3x}$$

$$= 0$$

$$= RS \quad \checkmark$$

Coefficient = 0 because 7 is a root of the auxiliary equation

Ex: Solve $2y'' - 24y' + 72y = 0$

auxiliary equation $2m^2 - 24m + 72 = 0$

$m^2 - 12m + 36 = 0$

$(m-6)^2 = 0$

$m = 6, 6$

$y_1 = e^{6x}$, $y_2 = xe^{6x}$

General Solution $y = C_1 y_1 + C_2 y_2$

$y = C_1 e^{6x} + C_2 x e^{6x}$ ✓

or $y = e^{6x}(C_1 + C_2 x)$ ✓

Ex: Solve $y'' - 6y' + 13y = 0$

$m^2 - 6m + 13 = 0$

$m = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$m = \dfrac{6 \pm \sqrt{-16}}{2}$ ~~$\sqrt{16}\sqrt{-1}$~~ $4i$

$m = \dfrac{6 \pm 4i}{2}$

$m = 3 \pm 2i$

$\alpha = 3, \beta = 2$

$$y_1 = e^{\alpha x} \cos \beta x \;, \quad y_2 = e^{\alpha x} \sin \beta x$$

$$y_1 = e^{3x} \cos 2x \;, \quad y_2 = e^{3x} \sin 2x$$

$$y = C_1 y_1 + C_2 y_2$$

$$y = C_1 e^{3x} \cos 2x + C_2 e^{3x} \sin 2x \quad \checkmark$$

or $y = e^{3x}(C_1 \cos 2x + C_2 \sin 2x)$ $\checkmark$