3.1 First-Order Models Cont'd

Ex: A tank contains 400 L of water in which 50 g of salt is dissolved. Saltwater of concentration $2 \frac{g}{L}$ is pumped in at a rate of $8 \frac{L}{min}$. The well-mixed solution is pumped out at $8 \frac{L}{min}$. Find $A(t)$, the mass of salt in the tank at time t (in g).

Let A = mass of salt in tank (g)

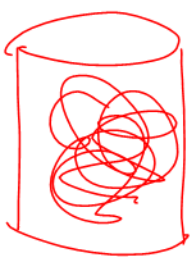
t = time (min)

$A(0) = 50$

Inflow Rate

$= 2 \frac{g}{L} \cdot 8 \frac{L}{min}$

$= 16 \frac{g}{min}$



Outflow Rate

$= \frac{A}{400} \frac{g}{L} \cdot 8 \frac{L}{min} = \frac{A}{50} \frac{g}{min}$

Note: $V = 400$ L is constant

$$\frac{dA}{dt} = \text{Inflow Rate} - \text{Outflow Rate}$$

$$\frac{dA}{dt} = 16 - \frac{A}{50}$$ **Linear**

$$\frac{dA}{dt} + \frac{A}{50} = 16$$ Standard Form

$$P(t) = \frac{1}{50}$$

$$\text{I.F.} = e^{\int \frac{1}{50}\,dt}$$

$$= e^{\frac{t}{50}}$$

$$e^{\frac{t}{50}}\frac{dA}{dt} + \frac{1}{50}e^{\frac{t}{50}}A = 16e^{\frac{t}{50}}$$

Integrate w.r.t. t:

$$e^{\frac{t}{50}}A = 16\left(50e^{\frac{t}{50}}\right) + C_1$$

$$A = 800 + C_1 e^{-\frac{t}{50}}$$

$A = 50$, $t = 0$: $\quad 50 = 800 + C_1$

$-750 = C_1$

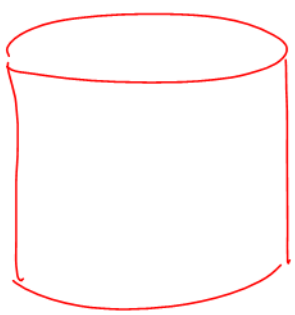
$A = 800 - 750e^{-\frac{t}{50}}$

Ex: A tank initially contains 500L of pure water. A solution of $2 \frac{kg}{L}$ salt flows in at $5 \frac{L}{min}$. Contents are mixed, and flow out at $10 \frac{L}{min}$. Find a formula for mass of salt (in kg) in tank at time t.

A = mass of salt in tank (in kg)

t = time (min)

$A(0) = 0$



Inflow Rate

$= 2 \frac{kg}{L} \cdot 5 \frac{L}{min}$

$= 10 \frac{kg}{min}$

$$\text{Volume} = 500 - 5t$$

$$\frac{dV}{dt} = \text{Inflow} - \text{Outflow}$$

$$= 5 - 10$$

$$= -5 \quad \frac{L}{min}$$

$$V = -5t + C_1$$

$V(0) = 500$:

$$500 = C_1$$

$$V = 500 - 5t$$

$$\text{Outflow Rate} = \frac{A}{500 - 5t} \frac{kg}{L} \cdot 10 \frac{L}{min}$$

$$= \frac{10A}{500 - 5t}$$

$$= \frac{2A}{100 - t}$$

$$\frac{dA}{dt} = \text{Inflow Rate} - \text{Outflow Rate}$$

$$\frac{dA}{dt} = 10 - \frac{2A}{100 - t}$$

Not separable

Linear

$$\frac{dA}{dt} + \frac{2}{100-t}A = 10$$ Standard Form

ASIDE

Not linear $\frac{dA}{dt} + A^{(2)} = 10$

Not linear $\frac{dA}{dt} + tA = \sin A$

$$P(t) = \frac{2}{100-t}$$

$$I.F. = e^{\int \frac{2}{100-t}\,dt}$$

$u = 100-t$

$du = -dt$

$dt = -du$

$$\int \frac{2}{100-t}\,dt = \int \frac{-2\,du}{u}$$

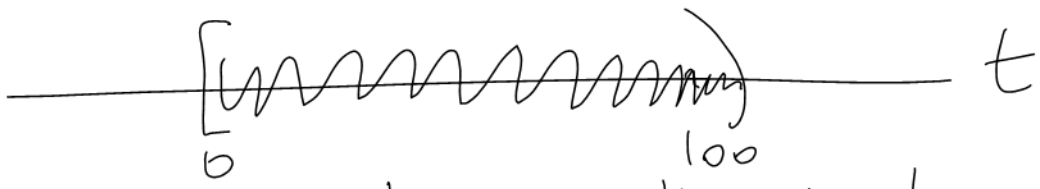
$$= -2\ln|u| + C_1$$

$$= e^{-2\ln|100-t|}$$

$$= e^{\ln(100-t)^{-2}}$$

$$= (100-t)^{-2}$$

Interval of Solution:



physics constraint math constraint

Interval: $0 \le t < 100$

$$(100-t)^{-2}\frac{dA}{dt} + 2(100-t)^{-3}A = 10(100-t)^{-2}$$

Integrate w.r.t. t:

$$(100-t)^{-2}A = 10(100-t)^{-1} + C_1$$

Same sub
$u = 100-t$

$\int 10(100-t)^{-2}\,dt$
$= \int -10u^{-2}\,du$
$= 10u^{-1} + C_1$

$$A = 10(100-t) + C_1(100-t)^2$$

$A=0$, $t=0$: $0 = 10(100) + C_1(10000)$

$C_1 = -0.1$

$$A = 10(100-t) - 0.1(100-t)^2$$

Follow-up:

$A(20)$? $A(20) = 160$ kg

$A(100)$? $A(100) = 0$

Still makes sense physically (tank is empty).