3.1 First-Order Models Cont'd

Newton's Law of Cooling:
The rate at which an object warms/cools is proportional to the temperature difference between the object and its environment.

Let T = temperature (°C)
t = time (minutes)
T_m = ambient temperature (°C) (constant)

$$\frac{dT}{dt} = k(T - T_m) \qquad k: \text{constant}$$

Ex: Room temp = 20°C
Initially coffee is 90°C.
After 40 mins coffee is 30°C.
Time when coffee was 60°C?

T = coffee's temp (°C)
t = time (mins)
$T(0) = 90$, $T(40) = 30$

$$\frac{dT}{dt} = k(T - 20) \qquad k: \text{constant}$$

$$\frac{dT}{T-20} = k\,dt \qquad \text{Separable}$$

$$\int \frac{dT}{T-20} = \int k\,dt$$

$$\ln|T-20| = kt + C_1$$

Recall:

$$\int \frac{dx}{ax+b} = \frac{1}{a}\ln|ax+b| + C_1$$

$$|T-20| = e^{kt+C_1}$$

$$T-20 = \pm e^{C_1} e^{kt}$$

$$T-20 = Ce^{kt}$$

$$\boxed{T = 20 + Ce^{kt}}$$

$T=90$, $t=0$:

$$90 = 20 + C$$

$$70 = C$$

$$\boxed{T = 20 + 70e^{kt}}$$

$T=30$, $t=40$:

$$30 = 20 + 70e^{40k}$$

$$10 = 70e^{40k}$$

$$\frac{1}{7} = e^{40k}$$

$$\ln\left(\frac{1}{7}\right) = 40k$$

$$\frac{1}{40}\ln\left(\frac{1}{7}\right) = k$$

$$T = 20 + 70e^{\frac{1}{40}\ln\left(\frac{1}{7}\right)t}$$

$T = 60$:

$$60 = 20 + 70e^{\frac{1}{40}\ln\left(\frac{1}{7}\right)t}$$

$$40 = 70e^{\frac{1}{40}\ln\left(\frac{1}{7}\right)t}$$

$$\frac{4}{7} = e^{\frac{1}{40}\ln\left(\frac{1}{7}\right)t}$$

$$\ln\left(\frac{4}{7}\right) = \frac{1}{40}\ln\left(\frac{1}{7}\right)t$$

$$\frac{40\ln\left(\frac{4}{7}\right)}{\ln\left(\frac{1}{7}\right)} = t$$

$$t \approx 12 \text{ minutes}$$

3.1 #16

Container A is maintained at 0°C.
" B " 100°C.

Metal bar, initially 100°C, is placed in Container A for 2 minutes. After 1 min. in Container A the bar was 90°C.
Then the bar is immediately placed in Container B. After 1 min. in Container B the bar's temp. rises by 10°C.
How long, from the start of the process, does it take for the bar to reach 99.9°C?

Only need to solve the DE once.

$$\frac{dT}{dt} = k(T - T_m)$$

constants

$$\frac{dT}{T - T_m} = k\,dt$$

$$\int \frac{dT}{T - T_m} = \int k\,dt$$

$$\ln|T - T_m| = kt + C_1$$

$$T - T_m = Ce^{kt}$$

$$\boxed{T = T_m + Ce^{kt}}$$

Phase 1: Container A

$T(0) = 100$, $T(1) = 90$, $T_m = 0$

$T_m = 0$: $T = Ce^{kt}$

$t = 0$, $T = 100$: $T = 100e^{kt}$

$t = 1$, $T = 90$: $T = 100e^{(\ln 0.9)t}$

$t = 2$: $T = 81$

Phase 2: Container B

t = time in Container B (minutes)

$T(0) = 81$, $T(1) = 91$, $T_m = 100$

$$T = T_m + Ce^{kt}$$

$T_m = 100$: $T = 100 + Ce^{kt}$

$T = 81$, $t = 0$: $T = 100 - 19e^{kt}$

$T = 91$, $t = 1$: $T = 100 - 19e^{\ln\left(\frac{9}{19}\right)t}$

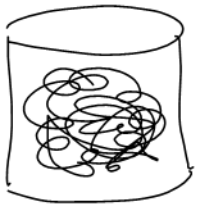
$T = 99.9$: $99.9 = 100 - 19e^{\ln\left(\frac{9}{19}\right)t}$

$\vdots$

$t \approx 7.02$ minutes

$$2 + 7.02 \approx 9.02 \text{ min}$$

Ex:



400 L of water
50 g salt

salt solution $\frac{2g}{L}$ @ $\frac{8L}{min}$

@ $\frac{8L}{min}$

To be continued