3.1 First-Order Models

Recall: "A is proportional to B"

means $A = kB$ where k is a constant.

Ex: Rate of population increase is proportional to the population size.
Population doubles in 5 years.
Time for the population to triple?

Let P = population size
t = time (years)
P_0 = initial population size

$P(0) = P_0$, $P(5) = 2P_0$

$$\frac{dP}{dt} = kP$$ separable

rate of change of population w.r.t. time

$$\frac{dP}{P} = k\,dt$$

$$\int \frac{dP}{P} = \int k\,dt$$

$$\ln|P| = kt + C_1$$

$e^{LS} = e^{RS}$:

$$|P| = e^{kt+C_1}$$

$$|P| = e^{C_1} e^{kt}$$

$$P = \pm e^{C_1} e^{kt}$$

C

$$\boxed{P = Ce^{kt}}$$

$P = P_0$, $t = 0$:

$$P_0 = C$$

$$\boxed{P = P_0 e^{kt}}$$

$P = 2P_0$, $t = 5$:

$$2P_0 = P_0 e^{5k}$$

$$2 = e^{5k}$$

$$\ln 2 = 5k$$

$$\frac{\ln 2}{5} = k$$

$$\boxed{P = P_0 e^{\frac{\ln 2}{5} t}}$$

$P = 3P_0$: $3P_0 = P_0 e^{\frac{\ln 2}{5} t}$

$$3 = e^{\frac{\ln 2}{5} t}$$

$$\ln 3 = \frac{\ln 2}{5} t$$

$$\frac{5 \ln 3}{\ln 2} = t$$

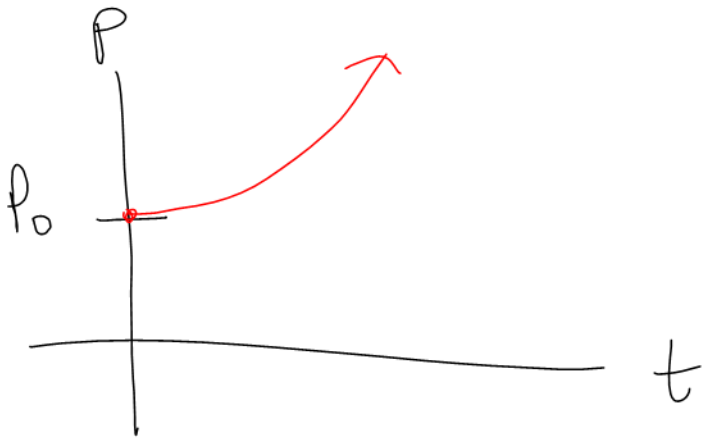
$$t \approx 8 \text{ years}$$

Notice: We did not need to know P_0. But it was helpful to have a symbol for it.

$P = P_0 e^{kt}$ with $k > 0$

is <u>exponential growth</u>.

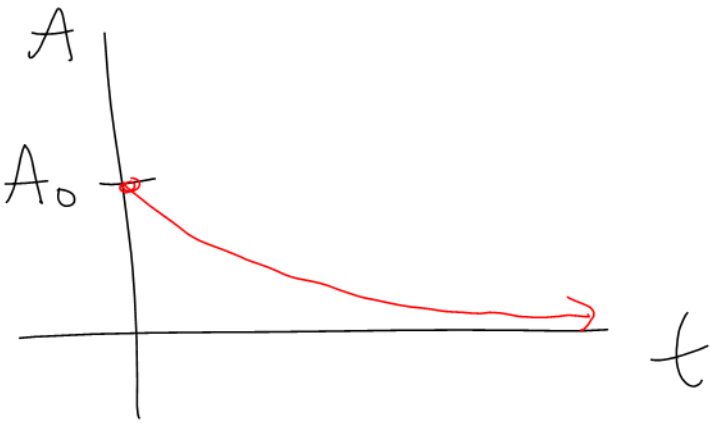
$P \to \infty$ as $t \to \infty$



$A = A_0 e^{kt}$ with $k < 0$

is <u>exponential decay</u>.

$A \to 0$ as $t \to \infty$



Ex: Rate of decay of a radioactive substance is proportional to the amount remaining. After 1000 years, 3.5% of the original amount has decayed. Find the substance's half-life.

let A = amount remaining

A_0 = initial amount

t = time (years)

Half-life: time for 50% of the substance to decay.

$$\frac{dA}{dt} = kA$$

rate of change of amount w.r.t. time

Separable

$$\frac{dA}{A} = k\,dt$$

$$\int \frac{dA}{A} = \int k\,dt$$

$$\ln|A| = kt + C_1$$

$$|A| = e^{kt+C_1}$$

$$A = \pm e^{C_1} e^{kt}$$

$$A = Ce^{kt}$$

$t = 0$

$A = A_0$:

$$A_0 = C$$

$$\boxed{A = A_0 e^{kt}}$$

3.5% decayed

$\Rightarrow$ 96.5% remaining

Sub $t = 1000$

$A = 0.965 A_0$:

$$0.965A_0 = A_0 e^{1000k}$$

$$0.965 = e^{1000k}$$

$$\ln 0.965 = 1000k$$

$$\frac{\ln 0.965}{1000} = k$$

$$\boxed{A = A_0 e^{\frac{\ln 0.965}{1000} t}}$$

Sub $A = 0.5A_0$:

$$0.5A_0 = A_0 e^{\frac{\ln 0.965}{1000} t}$$

$$0.5 = e^{\frac{\ln 0.965}{1000} t}$$

$$\ln 0.5 = \frac{\ln 0.965}{1000} t$$

$$\frac{1000 \ln 0.5}{\ln 0.965} = t$$

$$t \approx 19{,}000 \text{ years}$$

Note for Suggested HW

3.1 #11

Half-life of Carbon-14 is 5600 years.

Newton's Law of Cooling

Rate at which an object warms/cools is proportional to temperature difference between object and environment.