2.4 Exact DEs Cont'd

If a DE is not exact, we can sometimes find an I.F.

If $\frac{M_y - N_x}{N}$ is a function of x (no y-terms),

use I.F. $= e^{\int \frac{M_y - N_x}{N}\,dx}$.

If $\frac{N_x - M_y}{M}$ is a function of y (no x-terms),

use I.F. $= e^{\int \frac{N_x - M_y}{M}\,dy}$

Ex: Solve $(e^{2x} + y - 1)\,dx - dy = 0$

$M = e^{2x} + y - 1$ $\quad N = -1$

$M_y = 1 \quad N_x = 0$

DE is not exact.

$\frac{M_y - N_x}{N} = \frac{1-0}{-1} = -1$ is a function of x

I.F. $= e^{\int \frac{M_y - N_x}{N}\,dx}$

$= e^{\int -1\,dx}$

$= e^{-x}$

Multiply the DE by I.F.

$$(e^{x} + e^{-x}y - e^{-x})dx - e^{-x}dy = 0$$

DE is now exact.

$f = \int (e^{x} + e^{-x}y - e^{-x})dx$ AND $f = \int -e^{-x}dy$

$f = e^{x} - e^{-x}y + e^{-x} + g(y)$ $\qquad$ $f = -e^{-x}y + h(x)$

$h(x) = e^{x} + e^{-x}$, $\quad g(y) = 0$

$$\Rightarrow f = e^{x} + e^{-x} - e^{-x}y$$

DE: $df = 0$

$\int df = \int 0$

$f = C$

$$\boxed{e^{x} + e^{-x} - e^{-x}y = C}$$ ✓

or $\quad e^{2x} + 1 - y = Ce^{x}$ ✓

$e^{2x} + 1 + C_1 e^{x} = y$ ✓

When faced with a DE, check in this order:

Is it separable?

Is it linear?

Is it exact?

Can we find an I.F. to make it exact?

2.5 Solutions by Substitution

Ex: $(y^2 + yx)\,dx + x^2\,dy = 0$

each term has degree 2

"DE is homogeneous of degree 2"

Ex: $(3x^2y - y^3)\,dx - 7x^3\,dy = 0$

DE is homogeneous of degree 3

Sub $\begin{cases} y = ux \\ dy = u\,dx + x\,du \end{cases}$ or $\begin{cases} x = vy \\ dx = v\,dy + y\,dv \end{cases}$

ASIDE

$\frac{dy}{dx} = u + x\frac{du}{dx}$

$dy = u\,dx + x\,du$

ASIDE

$\frac{dx}{dy} = v + y\frac{dv}{dy}$

$dx = v\,dy + y\,dv$

Comment: Sub $y = ux$ when the dy term is simpler than the dx term.

Comment: After the substitution, the DE becomes <u>separable</u>.

<u>Ex</u>: Solve explicitly

$$(x^2 + y^2)\,dx - xy\,dy = 0.$$

DE is homogeneous of degree 2

dy term is simpler:

Sub $\begin{cases} y = ux \\ dy = u\,dx + x\,du \end{cases}$

$$(x^2 + u^2x^2)\,dx - \underbrace{x(ux)}_{ux^2}(u\,dx + x\,du) = 0$$

$$x^2\,dx + u^2x^2\,dx - u^2x^2\,dx - ux^3\,du = 0$$

$$x^2\,dx = ux^3\,du \quad \text{Separable}$$

$$\frac{dx}{x} = u\,du$$

$$\int \frac{dx}{x} = \int u\,du$$

$$\ln|x| = \frac{u^2}{2} + C_1$$

$y = ux$

$\frac{y}{x} = u$

$$\ln|x| = \frac{1}{2}\left(\frac{y}{x}\right)^2 + C_1$$

$$2\ln|x| = \left(\frac{y}{x}\right)^2 + C_2$$

$$2\ln|x| + C_3 = \left(\frac{y}{x}\right)^2$$

$$\pm\sqrt{2\ln|x| + C_3} = \frac{y}{x}$$

$$y = \pm x\sqrt{2\ln|x| + C}$$

Ex: Solve $(x + ye^{y/x})dx - xe^{y/x}dy = 0$, $y(1) = 0$. Do not simplify the solution.

Comment: Ignore e^x, $\sin x$, $\ln x$ etc. when calculating degrees.

$\sqrt{x} = x^{1/2}$ has degree $\frac{1}{2}$.

DE is homogeneous of degree 1.

dy term is simpler.

Sub $\begin{cases} y = ux \\ dy = u\,dx + x\,du \end{cases}$

$\frac{y}{x} = u$

$$(x + uxe^u)dx - xe^u(u\,dx + x\,du) = 0$$

$$x\,dx + uxe^u dx - uxe^u dx - x^2 e^u du = 0$$

$$x\,dx = x^2 e^u du \qquad \text{Separable}$$

$$\frac{dx}{x} = e^u du$$

$$\int \frac{dx}{x} = \int e^u du$$

$$\ln|x| = e^u + C_1$$

$$\ln|x| = e^{y/x} + C_1$$

Sub $y = 0$, $x = 1$:

$$0 = 1 + C_1$$

$$-1 = C_1$$

$$\boxed{\ln|x| = e^{y/x} - 1}$$