2.4 Exact DEs Cntd

The differential of $f(x,y)$ is

$$df = f_x\,dx + f_y\,dy.$$

A DE $M\,dx + N\,dy = 0$ is exact if the left side is a differential.

Quick Way to Check:

$M\,dx + N\,dy = 0$ is exact if $M_y = N_x$.

Ex: Solve $\sin y\,dx + (2y + x\cos y)\,dy = 0$.

$M = \sin y \qquad N = 2y + x\cos y$

$M_y = \cos y \qquad N_x = \cos y$

$M_y = N_x \Rightarrow$ DE is exact

DE: $df = 0$

$\int df = \int 0$

Solution to DE: $f = C$

$f = \int \sin y\,dx$ AND $f = \int (2y + x\cos y)\,dy$

$f = x\sin y + \underbrace{g(y)}_{y\text{-terms}}$ AND $f = y^2 + x\sin y + \underbrace{h(x)}_{x\text{-terms}}$

$g(y) = y^2$ $\qquad$ $h(x) = 0$

$$\Rightarrow \quad f = x\sin y + y^2$$

DE : $\quad df = 0$

$$\int df = \int 0$$

Solution to DE $\quad f = C$

$$\boxed{x\sin y + y^2 = C}$$

Ex: Solve explicitly

$$(1+y^2)\,dx + (2xy + 2y)\,dy = 0$$

$$y(0) = 1$$

$M = 1+y^2 \qquad N = 2xy + 2y$

$M_y = 2y \qquad N_x = 2y$

$M_y = N_x \Rightarrow$ DE is exact

$f = \int (1+y^2)\,dx$ AND $f = \int (2xy + 2y)\,dy$

$f = x + xy^2 + g(y) \qquad f = xy^2 + y^2 + h(x)$

$$\Rightarrow \quad f = x + xy^2 + y^2$$

$$DE: \quad df = 0$$

$$\int df = \int 0$$

$$f = C$$

$$\boxed{x + xy^2 + y^2 = C}$$

$y = 1$, $x = 0$:

$$0 + 0 + 1 = C$$

$$1 = C$$

$$x + xy^2 + y^2 = 1$$

$$xy^2 + y^2 = 1 - x$$

$$(x+1)\,y^2 = 1 - x$$

$$y^2 = \frac{1-x}{1+x}$$

$$y = \pm\sqrt{\frac{1-x}{1+x}}$$

But $y(0) = 1$: choose $\oplus$

$$y = \sqrt{\frac{1-x}{1+x}}$$

Ex: Solve $(e^{x+y}+2y)\,dy = -(e^{x+y}+3x^2)\,dx$

$M\,dx + N\,dy = 0$: $(e^{x+y}+3x^2)\,dx + (e^{x+y}+2y)\,dy = 0$

$(e^x e^y + 3x^2)\,dx + (e^x e^y + 2y)\,dy = 0$

M $\qquad$ N

$M_y = e^x e^y \qquad N_x = e^x e^y$

$M_y = N_x \Rightarrow$ DE is exact

$f = \int (e^x e^y + 3x^2)\,dx$ AND $f = \int (e^x e^y + 2y)\,dy$

$f = e^x e^y + x^3 + g(y)$ $\qquad$ $f = e^x e^y + y^2 + h(x)$

$\Rightarrow f = e^x e^y + x^3 + y^2$

Optional: Check $f = e^x e^y + x^3 + y^2$

$df = f_x\,dx + f_y\,dy$

$df = (e^x e^y + 3x^2)\,dx + (e^x e^y + 2y)\,dy$ ✓

DE: $df = 0$

$\int df = \int 0$

$f = C$

$$e^x e^y + x^3 + y^2 = C$$

If the DE is not exact, we can sometimes find an integrating factor.

If $\frac{M_y - N_x}{N}$ is a function of x (no y-terms),

use $\text{I.F.} = e^{\int \frac{M_y - N_x}{N}\,dx}$

If $\frac{N_x - M_y}{M}$ is a function of y (no x-terms),

use $\text{I.F.} = e^{\int \frac{N_x - M_y}{M}\,dy}$.

(See formula sheet)

Ex: Solve $\underbrace{y}_{M}\,dx + \underbrace{(2x - ye^y)}_{N}\,dy = 0$

$M_y = 1 \qquad N_x = 2$

$M_y \neq N_x \Rightarrow$ DE is not exact

$\frac{M_y - N_x}{N} = \frac{1-2}{2x - ye^y}$ not a function of x ☹

$\frac{N_x - M_y}{M} = \frac{2-1}{y} = \frac{1}{y}$ a function of y ☺

$$\text{I.F.} = e^{\int \frac{1}{y}\,dy}$$

$$= e^{\ln|y|}$$

$$= |y|$$

$$= \begin{cases} y, & y>0 \\ -y, & y<0 \end{cases}$$

$$= y$$

Interval of solution: $y>0$

Multiply original DE by y:

$$y^2\,dx + (2xy - y^2e^y)\,dy = 0$$

Now an exact DE

$$f = \int y^2\,dx \qquad \text{AND} \qquad f = \int (2xy - y^2e^y)\,dy$$

$$f = xy^2 + g(y) \qquad f = xy^2 + (-y^2+2y-2)e^y + h(x)$$

$$\Rightarrow f = xy^2 + (-y^2+2y-2)e^y$$

	D	I
⊕	$-y^2$	e^y
⊖	$-2y$	
⊕	-2	
	0	

DE : $df = 0$

$\int df = \int 0$

Solution of DE : $f = C$

$$\boxed{xy^2 + (-y^2 + 2y - 2)e^y = C}$$