2.2 Separable Variables Cont'd

Ex: Find an explicit solution (solve for y). Interval of solution?

$$y' - ty^3 = 0, \quad y(0) = 2$$

$$\frac{dy}{dt} - ty^3 = 0$$

$$\frac{dy}{dt} = ty^3$$

$$y^{-3}\,dy = t\,dt$$

$$\int y^{-3}\,dy = \int t\,dt$$

$$-\frac{1}{2}y^{-2} = \frac{t^2}{2} + C_1$$

Sub $y=2$, $t=0$: $-\frac{1}{2}\left(\frac{1}{4}\right) = C_1$

$$C_1 = -\frac{1}{8}$$

$$-\frac{1}{2}y^{-2} = \frac{t^2}{2} - \frac{1}{8}$$

Multiply by -8: $4y^{-2} = -4t^2 + 1$

$$\frac{4}{1-4t^2} = y^2$$

$$y = \pm\sqrt{\frac{4}{1-4t^2}}$$

$$y = \pm\frac{2}{\sqrt{1-4t^2}}$$

But $y(0)=2 \Rightarrow$ Choose $+$

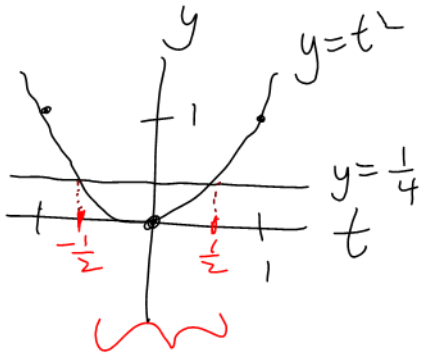
$$\boxed{y = \frac{2}{\sqrt{1-4t^2}}}$$

Interval of solution: $1-4t^2 > 0$

$$1 > 4t^2$$

$$\frac{1}{4} > t^2$$

$$-\frac{1}{2} < t < \frac{1}{2}$$



On tests, I'll indicate whether to give an explicit or implicit solution.

On suggested HW, they give explicit solutions if it's easy to do so, otherwise implicit.

2.3 Linear DEs

We'll solve $\frac{dy}{dx} + P(x)\,y = f(x)$

"1st order linear DE"

Recall: Product Rule

$$\frac{d}{dx}\left[e^{x^2} y\right] = e^{x^2}\frac{dy}{dx} + 2xe^{x^2} y$$

$$\int \left[e^{x^2}\frac{dy}{dx} + 2xe^{x^2} y\right]dx = e^{x^2} y + \text{constant}$$

Shortcut

Ex: Solve explicitly (solve for y)

$$y' + 2xy - x = 0$$

1) Standard Form: $\frac{dy}{dx} + P(x)\,y = f(x)$

$$\frac{dy}{dx} + 2xy - x = 0$$

$$\frac{dy}{dx} + 2xy = x \quad \text{standard form}$$

Left side does not come from the Product Rule

2) $P(x) = 2x$

3) Integrating Factor

$$\text{I.F.} = e^{\int P(x)\,dx}$$

$$= e^{\int 2x\,dx}$$

$$= e^{x^2}$$ ← Don't use a constant here.

4) Multiply the standard form by the I.F.

$$e^{x^2}\frac{dy}{dx} + 2xe^{x^2}y = xe^{x^2}$$

Left side comes from the Product Rule

5) Integrate with respect to x

(Left side comes from the product rule; use shortcut)

$$e^{x^2}y = \frac{1}{2}e^{x^2} + C_1$$

$u = x^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$\int xe^{x^2}dx = \int \frac{1}{2}e^{u}\,du$$

$$= \frac{1}{2}e^{u} + C_1$$

$$= \frac{1}{2}e^{x^2} + C_1$$

6) Explicit solution and Use the Initial Condition if Applicable

$$y = \frac{1}{2} + Ce^{-x^2}$$

Ex: Solve explicitly

$$x\frac{dy}{dx} - y - 2x^2 = 0, \quad y(1) = 5$$

$$x\frac{dy}{dx} - y = 2x^2$$

$$\frac{dy}{dx} - \frac{1}{x}y = 2x \quad \text{standard form}$$

$$P(x) = -\frac{1}{x}$$

$$I.F. = e^{\int P(x)\,dx}$$

$$= e^{\int -\frac{1}{x}\,dx}$$

$$= e^{-\ln|x|}$$

$$= e^{\ln |x|^{-1}}$$

$$= |x|^{-1}$$

$$= \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{-1}{x}, & x < 0 \end{cases}$$

$$= \frac{1}{x}$$

Interval of Solution: $x > 0$

$$\frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = 2$$

Integrate w.r.t. x:

$$\frac{1}{x}y = 2x + C_1$$

$$y = 2x^2 + Cx$$

$y = 5$, $x = 1$: $\quad 5 = 2 + C$

$$C = 3$$

$$\boxed{y = 2x^2 + 3x}$$

Ex: Solve explicitly

$$(x^2-16)\frac{dy}{dx} + xy - x = 0$$

$$(x^2-16)\frac{dy}{dx} + xy = x$$

$$\frac{dy}{dx} + \frac{x}{x^2-16}y = \frac{x}{x^2-16}$$

standard Form

$$P(x) = \frac{x}{x^2-16}$$

$$I.F. = e^{\int P(x)\,dx}$$

$$= e^{\int \frac{x}{x^2-16}\,dx}$$

$$= e^{\frac{1}{2}\ln|x^2-16|}$$

(No constant)

Sub $u = x^2-16$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$\int \frac{x\,dx}{x^2-16} = \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|u| + C_1$$

$$= e^{\ln |x^2-16|^{1/2}}$$

$$= \sqrt{x^2-16}$$