(11)

⋮

2) Solve for $Y(s)$

$$Y(s) = \frac{s+2}{s^2+4} + \frac{3}{(s+1)(s^2+4)} + \frac{s}{(s^2+4)^2}$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{s}{s^2+4} + \frac{2}{s^2+4} + \frac{1}{4}\cdot\frac{4s}{(s^2+4)^2} + \frac{3}{(s+1)(s^2+4)} \right\}$$

4) Partial Fractions

$$\frac{3}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

⋮

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{s}{s^2+4} + \frac{2}{s^2+4} + \frac{1}{4}\cdot\frac{4s}{(s^2+4)^2} + \frac{3}{5}\cdot\frac{1}{s+1} - \frac{3}{5}\cdot\frac{s}{s^2+4} + \underbrace{\frac{3}{5}\cdot\frac{1}{s^2+4}}_{\frac{3}{10}\cdot\frac{2}{s^2+4}} \right\}$$

$$y(t) = \cos 2t + \sin 2t + \frac{t\sin 2t}{4} + \frac{3}{5}e^{-t} - \frac{3}{5}\cos 2t + \frac{3}{10}\sin 2t \quad \checkmark$$

OR $y(t) = \frac{2}{5}\cos 2t + \frac{13}{10}\sin 2t + \frac{t\sin 2t}{4} + \frac{3}{5}e^{-t}$ ✓

(12) Solve $y'' + 4y' + 13y = 0$,

$y(0) = 7, \quad y'(0) = 3$

1) Apply $\mathcal{L}$

$$s^2Y(s) - sy(0) - y'(0) + 4[sY(s) - y(0)] + 13Y(s) = 0$$

$$s^2Y(s) - 7s - 3 + 4sY(s) - 28 + 13Y(s) = 0$$

2) Solve for $Y(s)$

$$[s^2 + 4s + 13]Y(s) = 7s + 31$$

$$Y(s) = \frac{7s + 31}{s^2 + 4s + 13}$$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{ \frac{7s + 31}{s^2 + 4s + 13} \right\} \quad (\star)$$

4) Complete the Square

$$s^2 + 4s + 13 = (s+2)^2 + ?$$

$$= (s+2)^2 + 9$$

$$= (s+2)^2 + 3^2$$

$$7s + 31 = ?(s+2) + ?$$

$$= 7(s+2) + ?$$

$$= 7(s+2) + 17$$

5) Find $y(t)$

From ⋆ $y(t) = \mathcal{L}^{-1}\left\{\frac{7(s+2)+17}{(s+2)^2+3^2}\right\}$

$$y(t) = \mathcal{L}^{-1}\left\{7\cdot\frac{s+2}{(s+2)^2+3^2} + \frac{17}{3}\cdot\frac{3}{(s+2)^2+3^2}\right\}$$

$$y(t) = 7e^{-2t}\cos 3t + \frac{17}{3}e^{-2t}\sin 3t$$

(13) Try on your own..

(14) $\lambda = 2, 2$

because $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is upper triangular.

$\lambda = 2$: Solve $[A - 2I \mid \vec{0}]$

$$\begin{matrix} x_1 & x_2 \\ \end{matrix}$$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \text{ RREF}$$

$\uparrow$

$x_1 = t$

$x_2 = 0$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} t$$

Choose a nonzero $\vec{x}$: $\vec{K}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{p}$: Solve $[A-\lambda I]\vec{p} = \vec{k}_1$

$$[A-2I \mid \vec{k}_1]$$

p_1 p_2

$$\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

RREF

$p_1 = t$

$p_2 = 1$

$$\vec{p} = \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Choose a nonzero $\vec{p}$: $\vec{p} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$\vec{x}_1 = \vec{k}_1 e^{\lambda t} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}$$

$$\vec{x}_2 = (\vec{k}_1 t + \vec{p}\,) e^{\lambda t} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) e^{2t}$$

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} t \\ 1 \end{bmatrix} e^{2t}$$

(15) Solve $\vec{x}' = \begin{bmatrix} -1 & -5 \\ 5 & -1 \end{bmatrix} \vec{x}$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -1-\lambda & -5 \\ 5 & -1-\lambda \end{vmatrix} = 0$$

$$(-1-\lambda)^2 + 25 = 0$$

$$\lambda^2 + 2\lambda + 26 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{-100}}{2} \quad \leftarrow 10i$$

$$\lambda = -1 \pm 5i \qquad (\alpha = -1, \beta = 5)$$

$\lambda = -1 + 5i$: Solve $[A - \lambda I \mid \vec{0}]$

$$[A - (-1+5i) I \mid \vec{0}]$$

$$\begin{bmatrix} -5i & -5 & 0 \\ 5 & -5i & 0 \end{bmatrix}$$

$$\frac{R_2}{5} \quad \begin{bmatrix} 1 & -i & 0 \\ -5i & -5 & 0 \end{bmatrix}$$

$$R_1$$

$$\begin{matrix} x_1 & x_2 \end{matrix}$$

$$R_2 + 5iR_1 \quad \begin{bmatrix} 1 & -i & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

RREF

$x_2 = t$

$x_1 - ix_2 = 0 \Rightarrow x_1 = it$

$$\vec{x} = \begin{bmatrix} i \\ 1 \end{bmatrix} t$$

Choose a nonzero $\vec{x}$: $\vec{k}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$$\vec{k}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$\vec{B}_1 = \mathrm{Re}(\vec{k}_1)$ $\qquad$ $\vec{B}_2 = \mathrm{Im}(\vec{k}_1)$

Note: $\vec{B}_2$ is a real vector

$$\vec{X}_1 = e^{\alpha t}\left(\vec{B}_1 \cos\beta t - \vec{B}_2 \sin\beta t\right)$$

$$= e^{-t}\left(\begin{bmatrix}0\\1\end{bmatrix}\cos 5t - \begin{bmatrix}1\\0\end{bmatrix}\sin 5t\right)$$

$$= e^{-t}\begin{bmatrix}-\sin 5t\\ \cos 5t\end{bmatrix}$$

$$\vec{X}_2 = e^{\alpha t}\left(\vec{B}_1 \sin\beta t + \vec{B}_2 \cos\beta t\right)$$

$$= e^{-t}\left(\begin{bmatrix}0\\1\end{bmatrix}\sin 5t + \begin{bmatrix}1\\0\end{bmatrix}\cos 5t\right)$$

$$= e^{-t}\begin{bmatrix}\cos 5t\\ \sin 5t\end{bmatrix}$$

$$\vec{X} = C_1\vec{X}_1 + C_2\vec{X}_2$$

$$\vec{X} = e^{-t}\left(C_1\begin{bmatrix}-\sin 5t\\ \cos 5t\end{bmatrix} + C_2\begin{bmatrix}\cos 5t\\ \sin 5t\end{bmatrix}\right)$$