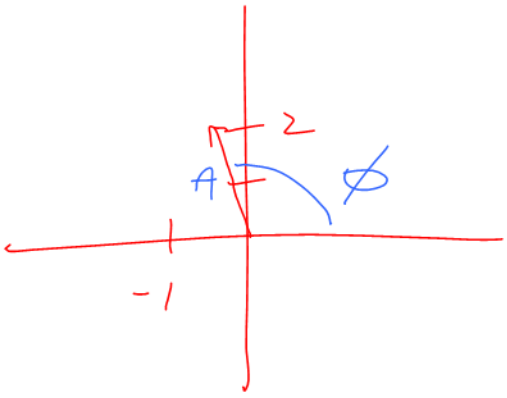
(8) a) $x = 2\cos 3t - \frac{1}{3}\sin 3t$

b) $x = 2\cos 3t - \frac{1}{3}\sin 3t$

$2 = A\sin\phi$; $-\frac{1}{3} = A\cos\phi$



$$A = \sqrt{2^2 + \left(-\frac{1}{3}\right)^2}$$

$$= \sqrt{4 + \frac{1}{9}}$$

$$= \sqrt{\frac{37}{9}}$$

$$= \frac{\sqrt{37}}{3}$$

$$\phi = \tan^{-1}\left(\frac{2}{\left(-\frac{1}{3}\right)}\right) \quad (+\pi\ ?)$$

$$= \tan^{-1}(-6) + \pi$$

$$\approx 1.74$$

$$x = A\sin(\omega t + \phi)$$

$$x = \frac{\sqrt{37}}{3}\sin(3t + 1.74)$$

(9)

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

DE: $(x+2)y'' + y = 0$

$$xy'' + 2y'' + y = 0$$

$$x\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + 2\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$$

$$+ \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-1} + \sum_{n=2}^{\infty} 2n(n-1) c_n x^{n-2}$$

$k = n-1$, $n = k+1$, starts at $k=1$

$k = n-2$, $n = k+2$, starts at $k=0$

$$+ \sum_{n=0}^{\infty} c_n x^n = 0$$

$k = n$, starts at $k=0$

Start at the largest k-value ($k=1$)

$$\sum_{k=1}^{\infty}(k+1)k\,C_{k+1}x^k + \underbrace{4C_2}_{\substack{k=0 \text{ term} \\ (1\text{st term})}} + \sum_{k=1}^{\infty}2(k+2)(k+1)C_{k+2}x^k$$

$$+ \underbrace{C_0}_{\substack{k=0 \text{ term} \\ (1\text{st term})}} + \sum_{k=1}^{\infty}C_k x^k = 0$$

$$4C_2 + C_0 + \sum_{k=1}^{\infty}\left[(k+1)k\,C_{k+1} + 2(k+2)(k+1)C_{k+2} + C_k\right]x^k = 0$$

Set all coefficients $= 0$

C_0, C_1 : unknown

$$4C_2 + C_0 = 0 \implies C_2 = -\frac{C_0}{4}$$

$$(k+1)k\,C_{k+1} + 2(k+2)(k+1)C_{k+2} + C_k = 0$$

$$\implies C_{k+2} = \frac{-C_k - (k+1)k\,C_{k+1}}{2(k+2)(k+1)}, \quad k \geq 1$$

$k=1$:

$$C_3 = \frac{-C_1 - 2C_2}{12}$$

$$= -\frac{C_1}{12} - \frac{C_2}{6}$$

$$= -\frac{C_1}{12} + \frac{C_0}{24}$$

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots$$

$$y = C_0 + C_1 x - \frac{C_0 x^2}{4} + \left(\frac{-C_1}{12} + \frac{C_0}{24}\right)x^3 + \cdots$$

$$y = C_0\underbrace{\left[1 - \frac{x^2}{4} + \cdots\right]}_{y_1} + C_1\underbrace{\left[x - \frac{x^3}{12} + \cdots\right]}_{y_2}$$

(10) $\begin{cases} \cos 2t, & 0 \le t < \pi \\ 0, & t \ge \pi \end{cases}$

$= ? + ? u(t-\pi)$

$= \cos 2t - (\cos 2t) u(t-\pi)$

DE: $y'' + 4y = \cos 2t - (\cos 2t) u(t-\pi)$

1) Apply $\mathcal{L}$

$s^2 Y(s) - s y(0) - y'(0) + 4Y(s) = \frac{s}{s^2+4} - e^{-\pi s} \mathcal{L}\{f(t-\pi)\}$

$\mathcal{L}\{f(t+\pi)\}$
$= \mathcal{L}\{\cos 2(t+\pi)\}$
$= \mathcal{L}\{\cos(2t+2\pi)\}$
$= \mathcal{L}\{\cos 2t \cos 2\pi - \sin 2t \sin 2\pi\}$
$= \mathcal{L}\{\cos 2t\}$
$= \frac{s}{s^2+4}$

$s^2 Y(s) - 1 + 4Y(s) = \frac{s}{s^2+4} - e^{-\pi s} \frac{s}{s^2+4}$

2) Solve for $Y(s)$

$(s^2+4)Y(s) = 1 + \frac{s}{s^2+4} - \frac{s}{s^2+4} e^{-\pi s}$

$Y(s) = \frac{1}{s^2+4} + \frac{s}{(s^2+4)^2} - \frac{s}{(s^2+4)^2} e^{-\pi s}$

3) Apply $\mathcal{L}^{-1}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+4} + \frac{s}{(s^2+4)^2} - \frac{s}{(s^2+4)^2}e^{-\pi s}\right\}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{1}{2}\cdot\frac{2}{s^2+4} + \frac{1}{4}\cdot\frac{4s}{(s^2+4)^2} - \frac{1}{4}\cdot\frac{4s}{(s^2+4)^2}e^{-\pi s}\right\}$$

$$y(t) = \frac{1}{2}\sin 2t + \frac{1}{4}t\sin 2t - f(t-\pi)\,u(t-\pi)$$

$$F(s) = \frac{1}{4}\cdot\frac{4s}{(s^2+4)^2}$$

$$f(t) = \frac{1}{4}t\sin 2t$$

$$f(t-\pi) = \frac{1}{4}(t-\pi)\sin 2(t-\pi)$$

$$y(t) = \frac{1}{2}\sin 2t + \frac{1}{4}t\sin 2t - \frac{1}{4}(t-\pi)\underline{\sin(2(t-\pi))}\,u(t-\pi)$$ ✓

OR

$$y(t) = \frac{1}{2}\sin 2t + \frac{1}{4}t\sin 2t - \frac{1}{4}(t-\pi)\underline{\sin 2t}\,u(t-\pi)$$ ✓

(Using $\sin(A+B) = \sin A\cos B + \cos A\sin B$)

(11) $y'' + 4y = 3e^{-t} + \cos 2t$

1) Apply $\mathcal{L}$

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = \frac{3}{s+1} + \frac{s}{s^2+4}$$

$$s^2Y(s) - s - 2 + 4Y(s) = \frac{3}{s+1} + \frac{s}{s^2+4}$$

2) Solve for $Y(s)$

$$(s^2+4)Y(s) = s+2 + \frac{3}{s+1} + \frac{s}{s^2+4}$$

$$Y(s) = \frac{s+2}{s^2+4} + \frac{3}{(s+1)(s^2+4)} + \frac{s}{(s^2+4)^2}$$

3) Apply $\mathcal{L}^{-1}$

⋮

4) Partial Fractions

⋮